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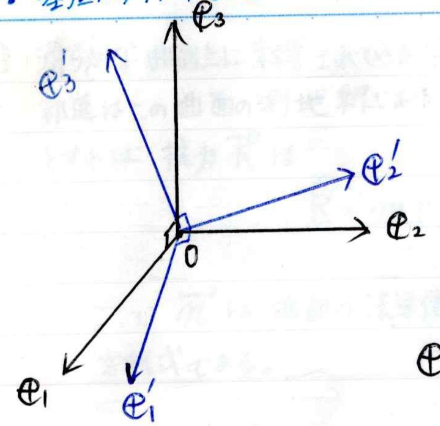
・基底ベクトルの変換



基底ベクトルの変換

$$\mathbf{e}_1' = a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2 + a_{31}\mathbf{e}_3$$
$$\mathbf{e}_2' = a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + a_{32}\mathbf{e}_3$$
$$\mathbf{e}_3' = a_{13}\mathbf{e}_1 + a_{23}\mathbf{e}_2 + a_{33}\mathbf{e}_3$$

$$\mathbf{e}_j' = \sum_{k=1}^{3} a_{kj}\mathbf{e}_k \quad —①$$

$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$　$\mathbf{e}_i$ は正規直交系をなすので

$\mathbf{e}_i' \mathbf{e}_j' = \delta_{ij}$　同様に $\mathbf{e}_i'$ も正規直交系をなすとすれば

$$\mathbf{e}_i' \cdot \mathbf{e}_j' = \sum_{k=1}^{3} a_{ki}\mathbf{e}_k \sum_{l=1}^{3} a_{lj}\mathbf{e}_l = \sum_{k=1}^{3}\sum_{l=1}^{3} a_{ki}a_{lj}\mathbf{e}_k\mathbf{e}_l$$
$$= \sum_{k=1}^{3}\sum_{l=1}^{3} a_{ki}a_{lj}\delta_{kl}$$
$$= \sum_{k=1}^{3} a_{ki}a_{kj} = \delta_{ij} \quad —②$$

a_{ij} が実数ならば　転置共役な行列をつくるとき
$(a_{ij} = \bar{a}_{ij})$

$$a_{ki} = a^*_{ik} \quad —③ \text{ なので} \qquad a^*_{ik} = {}^t\bar{a}_{ki} = {}^t a_{ki} = a_{ik}$$

$$\sum_{k=1}^{3} a^*_{ik} a_{kj} = \delta_{ij} \text{ とかける } —④$$

このことは

$\mathbf{e}_1'$の方向余弦を示す.　$\mathbf{e}_2'$の方向余弦　$\mathbf{e}_3'$の方向余弦

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$
が ユニタリー行列であることを示す。(直交行列)

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

②が示すことは次の様な行列の積となる。

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad —⑤$$

$$\sum_{k=1}^{3} x_k' \mathbf{e}_k' = \sum_{j=1}^{3} x_j \mathbf{e}_j \text{ とおくと } —⑥$$

$$\mathbf{e}_i' \cdot \sum_{k=1}^{3} x_k' \mathbf{e}_k' = \mathbf{e}_i' \sum_{j=1}^{3} x_j \mathbf{e}_j$$
$$= \sum_{l=1}^{3} a_{li}\mathbf{e}_l \sum_{j=1}^{3} x_j \mathbf{e}_j$$
$$= \sum_{l=1}^{3}\sum_{j=1}^{3} a_{li} x_j \mathbf{e}_l \cdot \mathbf{e}_j$$
$$= \sum_{l=1}^{3}\sum_{j=1}^{3} a_{li} x_j \delta_{lj}$$
$$x_i' = \sum_{l=1}^{3} a_{li} x_l = \sum_{j=1}^{3} a_{ji} x_j \quad —⑦$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad —⑧$$

$$\sum_{k=1}^{3} x'_k e'_k = \sum_{j=1}^{3} x_j e_j \quad ⑥$$

$$e_i \sum_{k=1}^{3} x'_k e'_k = e_i \sum_{j=1}^{3} x_j e_j$$

$$e_i \sum_{k=1}^{3} x'_k \sum_{l=1}^{3} a_{lk} e_l = x_i$$

$$\sum_{k=1}^{3} \sum_{l=1}^{3} a_{lk} x'_k e_i e_l = x_i$$

$$\therefore\ x_i = \sum_{k=1}^{3} \sum_{l=1}^{3} a_{lk} x'_k \delta_{il} = \sum_{k=1}^{3} a_{ik} x'_k \quad ⑨$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} \quad ⑩$$

⑧, ⑩ より

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} \quad ⑪$$

$$\therefore\ \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = A^t \cdot A \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} \quad ⑫$$

$$A^t \cdot A = E \quad ⑬$$

更に ⑧ ⑩ より

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = AA^t \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad ⑭$$

$$\therefore\ AA^t = E \quad ⑮$$

$$\therefore\ AA^t = A^t A = E \quad ⑯$$

$$\therefore\ A^t = A^{-1} \quad ⑰$$ 転置行列が逆行列である。

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} \quad ⑱$$

$$A^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} A \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} \quad ⑲$$

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = A^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^t \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad ⑳$$

∴ 基底ベクトルの変換により

$$\vec{\omega} = A \vec{\omega}'$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = A \begin{bmatrix} \omega'_1 \\ \omega'_2 \\ \omega'_3 \end{bmatrix} \quad ㉑$$

$$\vec{L} = A \vec{L}' \qquad \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = A \begin{bmatrix} L'_1 \\ L'_2 \\ L'_3 \end{bmatrix} \quad ㉒$$

◎ P16に詳細の展開あり

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \mathbf{I} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \quad —㉓ \qquad \mathbf{I} = \begin{bmatrix} I_{11} & -I_{12} & -I_{13} \\ -I_{21} & I_{22} & -I_{23} \\ -I_{31} & -I_{32} & I_{33} \end{bmatrix}$$

$$A \begin{bmatrix} L_1' \\ L_2' \\ L_3' \end{bmatrix} = \mathbf{I} A \begin{bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{bmatrix} \quad —㉔ \qquad -I_{12} = -I_{21}$$

実対称行列

$$A^{-1} A \begin{bmatrix} L_1' \\ L_2' \\ L_3' \end{bmatrix} = A^{-1} \mathbf{I} A \begin{bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{bmatrix} \quad —㉕$$

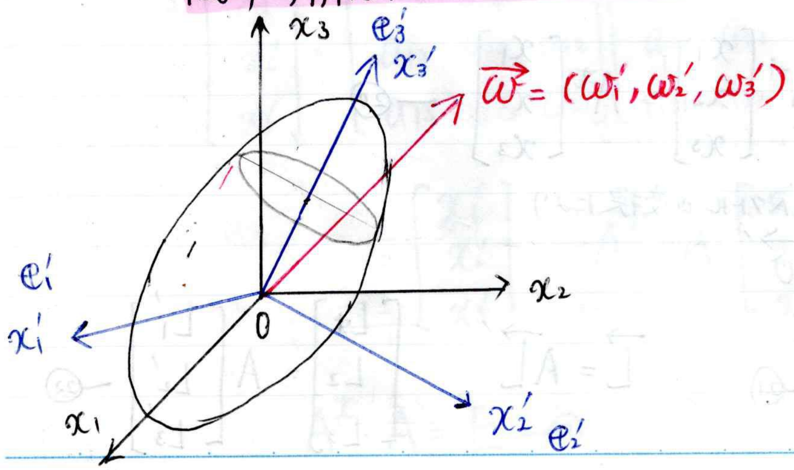
$$\begin{bmatrix} L_1' \\ L_2' \\ L_3' \end{bmatrix} = A^{-1} \mathbf{I} A \begin{bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{bmatrix} \quad —㉖$$

$$\begin{bmatrix} L_1' \\ L_2' \\ L_3' \end{bmatrix} = A^{t} \mathbf{I} A \begin{bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{bmatrix} \quad —㉗$$

剛体の

1点Oを固定した剛体の運動について

$\mathbf{I}$ が 実対称行列(エルミート行列)なので $\mathbf{I}$ は 適当な 直交行列 A (ユニタリ行列) により 対角化することが 可能である。



$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \boxed{\begin{aligned} e_1' &= a_{11} e_1 + a_{21} e_2 + a_{31} e_3 \\ e_2' &= a_{12} e_1 + a_{22} e_2 + a_{32} e_3 \\ e_3' &= a_{13} e_1 + a_{23} e_2 + a_{33} e_3 \end{aligned}}$$

$\mathbf{I}$ を 対角化する様な 剛体に固定した 座標系 e_1', e_2', e_3' を慣性の主軸と言う。

慣性主軸

主軸方向に 座標系をとるとき、剛体の 角運動量 $\vec{L}$, 回転の角速度 $\vec{\omega}$ を次の様にかくと

$$\vec{\omega} = \omega_1' e_1' + \omega_2' e_2' + \omega_3' e_3' \quad —㉘$$

$$\vec{L} = L_1' e_1' + L_2' e_2' + L_3' e_3' \quad —㉙ \quad は$$

㉗より

$$\begin{bmatrix} L_1' \\ L_2' \\ L_3' \end{bmatrix} = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{bmatrix}$$

$$L_1' = I_{11} \omega_1' \quad L_2' = I_{22} \omega_2' \quad L_3' = I_{33} \omega_3' \quad —㉚$$

$$\frac{d\vec{L}}{dt} = \vec{N} \quad は$$

$$\frac{d\vec{L}}{dt} = \frac{dL_1'}{dt} \cdot e_1' + \frac{dL_2'}{dt} e_2' + \frac{dL_3'}{dt} e_3 + L_1 \frac{de_1'}{dt} + L_2 \frac{de_2'}{dt} + L_3 \frac{de_3'}{dt} \quad —㉛$$

$$= \frac{dL_1'}{dt} \cdot e_1' + \frac{dL_2'}{dt} e_2' + \frac{dL_3'}{dt} e_3' + L_1' \vec{\omega} \times e_1' + L_2' \vec{\omega} \times e_2' + L_3' \vec{\omega} \times e_3'$$

$$= \frac{dL_1'}{dt} e_1' + \frac{dL_2'}{dt} e_2 + \frac{dL_3'}{dt} e_3' + \vec{\omega} \times \{ L_1 e_1' + L_2' e_2' + L_3 e_3' \} \quad —㉜$$

また
$$\vec{N} = N_1'\boldsymbol{e}_1' + N_2'\boldsymbol{e}_2' + N_3'\boldsymbol{e}_3' \quad \text{— ㉝}$$
とかければ

$$\frac{dL_1'}{dt}\cdot\boldsymbol{e}_1' + \frac{dL_2'}{dt}\boldsymbol{e}_2' + \frac{dL_3'}{dt}\boldsymbol{e}_3' + (\omega_1'\boldsymbol{e}_1' + \omega_2'\boldsymbol{e}_2' + \omega_3'\boldsymbol{e}_3') \times (L_1'\boldsymbol{e}_1' + L_2'\boldsymbol{e}_2' + L_3'\boldsymbol{e}_3')$$

㉜を計算づづけると

$$= \frac{dL_1'}{dt}\boldsymbol{e}_1' + \frac{dL_2'}{dt}\boldsymbol{e}_2' + \frac{dL_3'}{dt}\boldsymbol{e}_3' + \omega_1'\boldsymbol{e}_1' \times L_2'\boldsymbol{e}_2' + \omega_1'\boldsymbol{e}_1' \times L_3'\boldsymbol{e}_3'$$
$$+ \omega_2'\boldsymbol{e}_2' \times L_1'\boldsymbol{e}_1' + \omega_2'\boldsymbol{e}_2' \times L_3'\boldsymbol{e}_3'$$
$$+ \omega_3'\boldsymbol{e}_3' \times L_1'\boldsymbol{e}_1' + \omega_3'\boldsymbol{e}_3' \times L_2'\boldsymbol{e}_2'$$

$$= \frac{dL_1'}{dt}\boldsymbol{e}_1' + \frac{dL_2'}{dt}\boldsymbol{e}_2' + \frac{dL_3'}{dt}\boldsymbol{e}_3' + L_2'\omega_1'\boldsymbol{e}_3' - L_3'\omega_1'\boldsymbol{e}_2' + L_1'\omega_2'(-\boldsymbol{e}_3')$$
$$+ L_3'\omega_2'\boldsymbol{e}_1' + L_1'\omega_3'\boldsymbol{e}_2' + L_2'\omega_3'(-\boldsymbol{e}_1')$$

$$= \frac{dL_1'}{dt}\boldsymbol{e}_1' + \frac{dL_2'}{dt}\boldsymbol{e}_2' + \frac{dL_3'}{dt}\boldsymbol{e}_3' + (L_3'\omega_2' - L_2'\omega_3')\boldsymbol{e}_1' + (L_1'\omega_3' - L_3'\omega_1')\boldsymbol{e}_2'$$
$$+ (L_2'\omega_1' - L_1'\omega_2')\boldsymbol{e}_3' \quad \text{— ㉞}$$

$L_1' = I_{11}\omega_1' \quad L_2' = I_{22}\omega_2' \quad L_3' = I_{33}\omega_3'$ を㉞へ代入する

$\therefore$
$$\frac{d\vec{L}}{dt} = \frac{dL_1'}{dt}\boldsymbol{e}_1' + \frac{dL_2'}{dt}\boldsymbol{e}_2' + \frac{dL_3'}{dt}\boldsymbol{e}_3' + (I_{33}\omega_3'\omega_2' - I_{22}\omega_2'\omega_3')\boldsymbol{e}_1'$$
$$+ (I_{11}\omega_1'\omega_3' - I_{33}\omega_3'\omega_1')\boldsymbol{e}_2'$$
$$+ (I_{22}\omega_2'\omega_1' - I_{11}\omega_1'\omega_2')\boldsymbol{e}_3' \quad \text{— ㉞}$$

㉝と比較すると

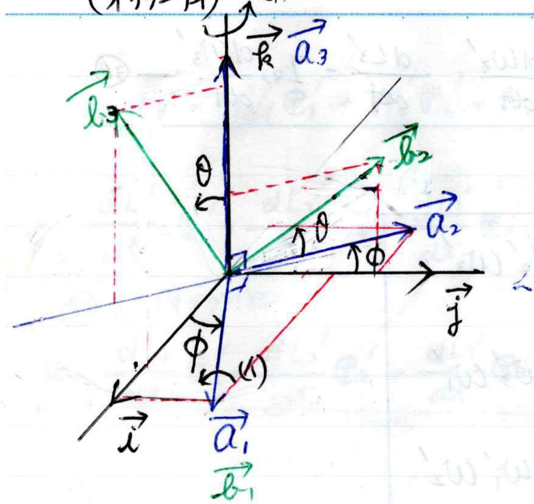
$$N_1' = \frac{dL_1'}{dt} + (I_{33} - I_{22})\omega_2'\omega_3'$$
$$N_2' = \frac{dL_2'}{dt} + (I_{11} - I_{33})\omega_1'\omega_3' \quad \text{— ㉟}$$
$$N_3' = \frac{dL_3'}{dt} + (I_{22} - I_{11})\omega_2'\omega_1'$$

$$\frac{dL_1'}{dt} = I_{11}\frac{d\omega_1'}{dt} \quad \frac{dL_2'}{dt} = I_{22}\frac{d\omega_2'}{dt} \quad \frac{dL_3'}{dt} = I_{33}\frac{d\omega_3'}{dt} \quad \text{— ㊱}$$

$$\boxed{\begin{aligned} N_1' &= I_{11}\frac{d\omega_1'}{dt} - (I_{22} - I_{33})\omega_2'\omega_3' \\ N_2' &= I_{22}\frac{d\omega_2'}{dt} - (I_{33} - I_{11})\omega_3'\omega_1' \\ N_3' &= I_{33}\frac{d\omega_3'}{dt} - (I_{11} - I_{22})\omega_1'\omega_2' \end{aligned}}$$

[オイラーの運動方程式]

(オイラー角)

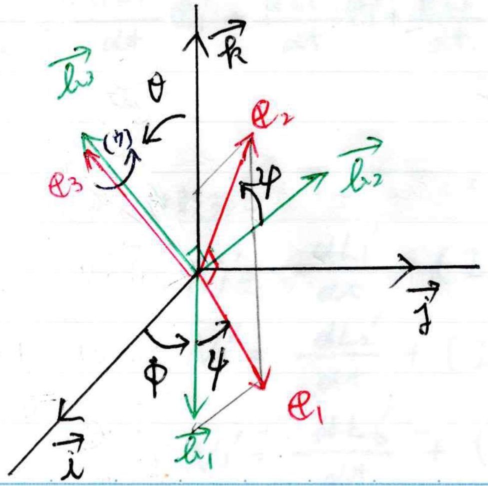


$$\begin{cases} \vec{a}_1 = \cos\phi\,\vec{i} + \sin\phi\,\vec{j} \\ \vec{a}_2 = -\sin\phi\,\vec{i} + \cos\phi\,\vec{j} \\ \vec{a}_3 = \vec{k} \end{cases}$$

$$\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} \quad \text{—①}$$

$$\begin{cases} \vec{b}_1 = \vec{a}_1 \\ \vec{b}_2 = \cos\theta\,\vec{a}_2 + \sin\theta\,\vec{a}_3 \\ \vec{b}_3 = -\sin\theta\cdot\vec{a}_2 + \cos\theta\,\vec{a}_3 \end{cases}$$

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}}_{C} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} \quad \text{—②}$$



$$e_1 = \cos\psi\,\vec{b}_1 + \sin\psi\,\vec{b}_2$$

$$e_2 = -\sin\psi\,\vec{b}_1 + \cos\psi\,\vec{b}_2$$

$$e_3 = \vec{b}_3$$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} \quad \text{—③}$$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} \quad \text{—④}$$

$$A = BCD \quad \text{—⑤}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\cos\theta\sin\phi & \cos\theta\cos\phi & \sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\cos\theta\sin\phi & \cos\theta\cos\phi & \sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi & \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix} \quad \text{—⑥}$$

$$\vec{\omega} = \dot{\phi}\vec{k} + \dot{\theta}\vec{a}_1 + \dot{\psi}\vec{b}_3 \quad \text{—⑦}$$

$$B^t = B^{-1} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{—⑧}$$

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \quad \text{—⑨}$$

$$\left.\begin{aligned} \vec{b}_1 &= \cos\psi\,\mathbf{e}_1 - \sin\psi\,\mathbf{e}_2 \\ \vec{b}_2 &= \sin\psi\,\mathbf{e}_1 + \cos\psi\,\mathbf{e}_2 \\ \vec{b}_3 &= \mathbf{e}_3 \end{aligned}\right\} \quad \text{—⑩}$$

同様に $A^t = A^{-1}$ —⑪

$$\begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} = A^t \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi & -\sin\psi\cos\phi - \cos\psi\,c \cdots in\phi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & -\sin\psi\sin\phi + \cos\psi \cdots \cos\phi & -\sin\theta\cos\phi \\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \quad \text{—⑫}$$

$$\vec{k} = \sin\psi\sin\theta\,\mathbf{e}_1 + \cos\psi\sin\theta\,\mathbf{e}_2 + \cos\theta\cdot\mathbf{e}_3 \quad \text{—⑬}$$

②より $\vec{b}_1 = \vec{a}_1$ $\quad\therefore\ \vec{a}_1 = \cos\psi\,\mathbf{e}_1 - \sin\psi\,\mathbf{e}_2$ —⑭

$$\therefore\ \vec{\omega} = \dot{\phi}\sin\psi\sin\theta\,\mathbf{e}_1 + \dot{\phi}\cos\psi\sin\theta\,\mathbf{e}_2 + \dot{\phi}\cos\theta\cdot\mathbf{e}_3 + \dot{\theta}\cos\psi\,\mathbf{e}_1 - \dot{\theta}\sin\psi\,\mathbf{e}_2 + \dot{\psi}\,\mathbf{e}_3 \quad \text{—⑮}$$

$$\vec{w} = (\dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi)\mathbf{e}_1 + (\dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi)\mathbf{e}_2 + (\dot{\phi}\cos\theta + \dot{\psi})\mathbf{e}_3 \quad \text{—⑯}$$

$$= \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 + \omega_3\mathbf{e}_3 \quad \text{—⑰}$$

$$\therefore\ \boxed{\begin{aligned} \omega_1 &= \dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi \\ \omega_2 &= \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi \\ \omega_3 &= \dot{\phi}\cos\theta + \dot{\psi} \end{aligned}} \quad \text{—⑱}$$