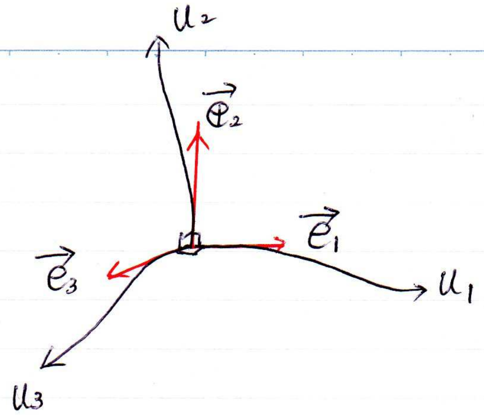
No. 2

Date 20 8 29



$h_1 = \sqrt{\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2}$ とおけば

$$dS_1 = \sqrt{\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2}\, du_1$$

$$\frac{dS_1}{du_1} = \sqrt{\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2} = h_1$$

$\vec{r}(x, y, z) \longrightarrow \vec{r}(u_1, u_2, u_3)$ と変数を変更する。

$$x = x(u_1, u_2, u_3) \quad y = y(u_1, u_2, u_3) \quad z = z(u_1, u_2, u_3)$$

$$dx = \frac{\partial x}{\partial u_1} du_1 \quad dy = \frac{\partial y}{\partial u_1} du_1 \quad dz = \frac{\partial z}{\partial u_1} du_1$$

$$\begin{aligned} dS_1^2 &= dx^2 + dy^2 + dz^2 \\ &= \left(\frac{\partial x}{\partial u_1}\right)^2 du_1^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 du_1^2 + \left(\frac{\partial z}{\partial u_1}\right)^2 du_1^2 \\ &= \left\{ \left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2 \right\} du_1^2 \end{aligned}$$

$$\therefore \ dS_1 = \sqrt{\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2}\, du_1 = h_1 du_1$$

$$\frac{\partial \vec{r}}{\partial u_1} = \left(\frac{\partial x}{\partial u_1}, \frac{\partial y}{\partial u_1}, \frac{\partial z}{\partial u_1} \right)$$

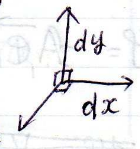
$$\left(\frac{\partial \vec{r}}{\partial u_1}\right)^2 = \left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2$$

$$\left|\frac{\partial \vec{r}}{\partial u_1}\right| = \sqrt{\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2} = h_1$$

また $d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 = \left(\frac{\partial x}{\partial u_1} du_1, \frac{\partial y}{\partial u_1} du_1, \frac{\partial z}{\partial u_1} du_1 \right)$

$$\overrightarrow{dr}^2 = \left(\frac{\partial x}{\partial u_1}\right)^2 du_1^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 du_1^2 + \left(\frac{\partial z}{\partial u_1}\right)^2 du_1^2$$

$$|\overrightarrow{dr}| = \sqrt{\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2}\, du_1$$



u_1 が増加する方向の単位ベクトル $\overrightarrow{e_1}$

$$\overrightarrow{e_1} = \frac{\dfrac{\partial \vec{r}}{\partial u_1}}{\left|\dfrac{\partial \vec{r}}{\partial u_1}\right|} = \frac{\dfrac{\partial \vec{r}}{\partial u_1}}{h_1}$$

$$h_2 = \sqrt{\left(\frac{\partial x}{\partial u_2}\right)^2 + \left(\frac{\partial y}{\partial u_2}\right)^2 + \left(\frac{\partial z}{\partial u_2}\right)^2} \qquad h_3 = \sqrt{\left(\frac{\partial x}{\partial u_3}\right)^2 + \left(\frac{\partial y}{\partial u_3}\right)^2 + \left(\frac{\partial z}{\partial u_3}\right)^2}$$

と定義すると

u_2, u_3 それぞれが増加する方向の単位ベクトル $\overrightarrow{e_2}$, $\overrightarrow{e_3}$ は

$$\overrightarrow{e_2} = \frac{\dfrac{\partial \vec{r}}{\partial u_2}}{\left|\dfrac{\partial \vec{r}}{\partial u_2}\right|} = \frac{\dfrac{\partial \vec{r}}{\partial u_2}}{h_2} \qquad \overrightarrow{e_3} = \frac{\dfrac{\partial \vec{r}}{\partial u_3}}{\left|\dfrac{\partial \vec{r}}{\partial u_3}\right|} = \frac{\dfrac{\partial \vec{r}}{\partial u_3}}{h_3}$$

$$dS_2 = \sqrt{\left(\frac{\partial x}{\partial u_2}\right)^2 + \left(\frac{\partial y}{\partial u_2}\right)^2 + \left(\frac{\partial z}{\partial u_2}\right)^2}\, du_2 = h_2 du_2$$

$$dS_3 = \sqrt{\left(\frac{\partial x}{\partial u_3}\right)^2 + \left(\frac{\partial y}{\partial u_3}\right)^2 + \left(\frac{\partial z}{\partial u_3}\right)^2}\, du_3 = h_3 du_3$$

$\overrightarrow{e_1}$, $\overrightarrow{e_2}$, $\overrightarrow{e_3}$ が　正規直交　ベクトルとすれば

$$\overrightarrow{e_1}\cdot\overrightarrow{e_2} = \overrightarrow{e_2}\cdot\overrightarrow{e_3} = \overrightarrow{e_3}\cdot\overrightarrow{e_1} = 0$$

$$\overrightarrow{e_1}^2 = \overrightarrow{e_2}^2 = \overrightarrow{e_3}^2 = 1$$

$$\overrightarrow{e_1} = \overrightarrow{e_2} \times \overrightarrow{e_3} \quad \overrightarrow{e_2} = \overrightarrow{e_3} \times \overrightarrow{e_1} \quad \overrightarrow{e_3} = \overrightarrow{e_1} \times \overrightarrow{e_2}$$

No. 4

Date 20 . 8 . 29

$\nabla\varphi = f_1\vec{e_1} + f_2\vec{e_2} + f_3\vec{e_3}$ と仮定する。—①

$\vec{r} = \vec{r}(u_1, u_2, u_3)$ とおければ

$d\vec{r} = \frac{\partial\vec{r}}{\partial u_1}du_1 + \frac{\partial\vec{r}}{\partial u_2}du_2 + \frac{\partial\vec{r}}{\partial u_3}du_3$ であり —②

前頁の関係式より $\left(\frac{\partial\vec{r}}{\partial u_1} = h_1\vec{e_1} \quad \frac{\partial\vec{r}}{\partial u_2} = h_2\vec{e_2} \quad \frac{\partial\vec{r}}{\partial u_3} = h_3\vec{e_3}\right)$ なので

$d\vec{r} = h_1du_1\vec{e_1} + h_2du_2\vec{e_2} + h_3du_3\vec{e_3}$ —③

$d\varphi = \nabla\varphi\cdot d\vec{r} = (f_1\vec{e_1} + f_2\vec{e_2} + f_3\vec{e_3})(h_1du_1\vec{e_1} + h_2du_2\vec{e_2} + h_3du_3$

$= f_1h_1du_1 + f_2h_2du_2 + f_3h_3du_3$ —④

全微分より

$d\varphi = \frac{\partial\varphi}{\partial u_1}du_1 + \frac{\partial\varphi}{\partial u_2}du_2 + \frac{\partial\varphi}{\partial u_3}du_3$ —⑤

④, ⑤を比較すると

$f_1h_1 = \frac{\partial\varphi}{\partial u_1} \quad f_2h_2 = \frac{\partial\varphi}{\partial u_2} \quad f_3h_3 = \frac{\partial\varphi}{\partial u_3}$ —⑥

$\therefore f_1 = \frac{1}{h_1}\frac{\partial\varphi}{\partial u_1} \quad f_2 = \frac{1}{h_2}\frac{\partial\varphi}{\partial u_2} \quad f_3 = \frac{1}{h_3}\frac{\partial\varphi}{\partial u_3}$ —⑦

$\therefore \nabla\varphi = \frac{1}{h_1}\frac{\partial\varphi}{\partial u_1}\vec{e_1} + \frac{1}{h_2}\frac{\partial\varphi}{\partial u_2}\vec{e_2} + \frac{1}{h_3}\frac{\partial\varphi}{\partial u_3}\vec{e_3}$ —⑧

$\nabla\varphi = \vec{e_1}\frac{1}{h_1}\frac{\partial\varphi}{\partial u_1} + \vec{e_2}\frac{1}{h_2}\frac{\partial\varphi}{\partial u_2} + \vec{e_3}\frac{1}{h_3}\frac{\partial\varphi}{\partial u_3}$ —⑨

$= \left(\vec{e_1}\frac{1}{h_1}\frac{\partial}{\partial u_1} + \vec{e_2}\frac{1}{h_2}\frac{\partial}{\partial u_2} + \vec{e_3}\frac{1}{h_3}\frac{\partial}{\partial u_3}\right)\varphi$ —⑩

$\varphi = u_1$ とおけば　(u_1, u_2, u_3) はそれぞれ独立変数なので

$$\nabla u_1 = \vec{e_1}\frac{1}{h_1}\frac{\partial u_1}{\partial u_1} = \frac{\vec{e_1}}{h_1} \qquad \vec{e_1} = h_1 \nabla u_1$$

$$\nabla u_2 = \vec{e_2}\frac{1}{h_2}\frac{\partial u_2}{\partial u_2} = \frac{\vec{e_2}}{h_2} \qquad \vec{e_2} = h_2 \nabla u_2$$

$$\nabla u_3 = \vec{e_3}\frac{1}{h_3}\frac{\partial u_3}{\partial u_3} = \frac{\vec{e_3}}{h_3} \qquad \vec{e_3} = h_3 \nabla u_3$$

$$\vec{A} = \nabla\varphi = \frac{\vec{e_1}}{h_1}\frac{\partial\varphi}{\partial u_1} + \frac{\vec{e_2}}{h_2}\frac{\partial\varphi}{\partial u_2} + \frac{\vec{e_3}}{h_3}\frac{\partial\varphi}{\partial u_3}$$ とおくと —⑪

$$\left[A_1 = \frac{1}{h_1}\frac{\partial\varphi}{\partial u_1} \quad A_2 = \frac{1}{h_2}\frac{\partial\varphi}{\partial u_2} \quad A_3 = \frac{1}{h_3}\frac{\partial\varphi}{\partial u_3} \right]$$

$$\nabla\vec{A} = \nabla(A_1\vec{e_1} + A_2\vec{e_2} + A_3\vec{e_3}) = \nabla(A_1\vec{e_1}) + \nabla(A_2\vec{e_2}) + \nabla(A_3\vec{e_3})$$ —⑫

$$\left.\begin{aligned} \vec{e_1} &= \vec{e_2}\times\vec{e_3} = (h_2\nabla u_2)\times(h_3\nabla u_3) = h_2h_3\nabla u_2\times\nabla u_3 \\ \vec{e_2} &= \vec{e_3}\times\vec{e_1} = (h_3\nabla u_3)\times(h_1\nabla u_1) = h_3h_1\nabla u_3\times\nabla u_1 \\ \vec{e_3} &= \vec{e_1}\times\vec{e_2} = (h_1\nabla u_1)\times(h_2\nabla u_2) = h_1h_2\nabla u_1\times\nabla u_2 \end{aligned}\right\}$$ —⑬

$$\nabla(A_1\vec{e_1}) = \nabla(A_1h_2h_3\nabla u_2\times\nabla u_3)$$

$$= \nabla(A_1h_2h_3)\cdot(\nabla u_2\times\nabla u_3) + A_1h_2h_3\nabla\cdot(\overset{A}{\nabla u_2}\times\overset{B}{\nabla u_3})$$

$$= \nabla(A_1h_2h_3)\cdot(\nabla u_2\times\nabla u_3) + Ah_2h_3\{\nabla u_3\cdot(\nabla\times\nabla u_2) - \nabla u_2(\nabla\times\nabla u_3)\}$$

$B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$ —⑭

$\nabla\times(\nabla\varphi) = 0$ より

$$\nabla(A_1\vec{e_1}) = \nabla(A_1h_2h_3)\cdot\frac{\vec{e_1}}{h_2h_3}$$ —⑮

—⑯

$$= \left(\vec{e_1}\frac{1}{h_1}\frac{\partial}{\partial u_1}(A_1h_2h_3) + \vec{e_2}\frac{1}{h_2}\frac{\partial(A_1h_2h_3)}{\partial u_2} + \vec{e_3}\frac{1}{h_3}\frac{\partial(A_1h_2h_3)}{\partial u_2}\right)\cdot\frac{\vec{e_1}}{h_2h_3}$$

$$\therefore\ \nabla(A_1\vec{e_1}) = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}(A_1h_2h_3) \quad \text{— (17)}$$

同様に

$$\nabla(A_2\vec{e_2}) = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_2}(A_2h_3h_1) \quad \text{— (18)}$$

$$\nabla(A_3\vec{e_3}) = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_3}(A_3h_1h_2) \quad \text{— (19)}$$

$\vec{A} = \nabla\varphi$ とおいたので

したがって $\nabla(A_1\vec{e_1} + A_2\vec{e_2} + A_3\vec{e_3}) = \nabla\cdot(\nabla\varphi) = \nabla^2\varphi$ — (20)

$$\therefore\ \nabla^2\varphi = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\varphi}{\partial u_1}\right) + \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\varphi}{\partial u_2}\right) + \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\varphi}{\partial u_3}\right) \quad \text{— (21)}$$

II
$$\nabla^2\varphi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\varphi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\varphi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\varphi}{\partial u_3}\right)\right]$$

I
$$\nabla\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(A_2h_3h_1) + \frac{\partial}{\partial u_3}(A_3h_1h_2)\right]$$

(一般的には)

III
$$\nabla\times\vec{A} = \frac{\vec{e_1}}{h_2h_3}\left[\frac{\partial}{\partial u_2}(A_3h_3) - \frac{\partial}{\partial u_3}(A_2h_2)\right] + \frac{\vec{e_2}}{h_3h_1}\left[\frac{\partial(A_1h_1)}{\partial u_3} - \frac{\partial}{\partial u_1}(A_3h_3)\right] + \frac{\vec{e_3}}{h_1h_2}\left[\frac{\partial}{\partial u_1}(A_2h_2) - \frac{\partial}{\partial u_2}(A_1h_1)\right]$$

$$\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\vec{e_1} & h_2\vec{e_2} & h_3\vec{e_3} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1h_1 & A_2h_2 & A_3h_3 \end{vmatrix}$$

行列式を用いた表現

$$\mathrm{rot}\,\vec{A} = \nabla\times(A_1\vec{e_1} + A_2\vec{e_2} + A_3\vec{e_3})$$

$$\nabla\times(A_1\vec{e_1}) = \qquad = \nabla\times(A_1 h_1 \nabla u_1)$$

$$= \nabla(A_1 h_1)\times\nabla u_1 + A_1 h_1 \nabla\times\nabla u_1$$

$$= \nabla(A_1 h_1)\times\nabla u_1$$

$$= \left(\frac{\vec{e_1}}{h_1}\frac{\partial}{\partial u_1}(A_1 h_1) + \frac{\vec{e_2}}{h_2}\frac{\partial}{\partial u_2}(A_1 h_1) + \frac{\vec{e_3}}{h_3}\frac{\partial (A_1 h_1)}{\partial u_3}\right)\times\frac{\vec{e_1}}{h_1}$$

$$= \frac{\vec{e_2}\times\vec{e_1}}{h_2 h_1}\frac{\partial}{\partial u_2}(A_1 h_1) + \frac{\vec{e_3}\times\vec{e_1}}{h_3 h_1}\frac{\partial (A_1 h_1)}{\partial u_3}$$

$$\nabla\times(A_1\vec{e_1}) = -\frac{\vec{e_3}}{h_1 h_2}\frac{\partial}{\partial u_2}(A_1 h_1) + \frac{\vec{e_2}}{h_3 h_1}\frac{\partial}{\partial u_3}(A_1 h_1)$$

$$= \frac{\vec{e_2}}{h_3 h_1}\frac{\partial}{\partial u_3}(A_1 h_1) - \frac{\vec{e_3}}{h_1 h_2}\frac{\partial}{\partial u_2}(A_1 h_1)$$

同様に

$$\nabla\times(A_2\vec{e_2}) = \frac{\vec{e_3}}{h_1 h_2}\frac{\partial}{\partial u_1}(A_2 h_2) - \frac{\vec{e_1}}{h_2 h_3}\frac{\partial}{\partial u_3}(A_2 h_2)$$

$$\nabla\times(A_3\vec{e_3}) = \frac{\vec{e_1}}{h_2 h_3}\frac{\partial}{\partial u_2}(A_3 h_3) - \frac{\vec{e_2}}{h_3 h_1}\frac{\partial}{\partial u_1}(A_3 h_3)$$

$$\nabla\times(A_1\vec{e_1} + A_2\vec{e_2} + A_3\vec{e_3})$$

$$= \left[\frac{1}{h_2 h_3}\frac{\partial}{\partial u_2}(A_3 h_3) - \frac{1}{h_2 h_3}\frac{\partial}{\partial u_3}(A_2 h_2)\right]\vec{e_1}$$

$$+ \left[\frac{1}{h_3 h_1}\frac{\partial}{\partial u_3}(A_1 h_1) - \frac{1}{h_3 h_1}\frac{\partial}{\partial u_1}(A_3 h_3)\right]\vec{e_2}$$

$$+ \left[\frac{1}{h_1 h_2}\frac{\partial}{\partial u_1}(A_2 h_2) - \frac{1}{h_1 h_2}\frac{\partial}{\partial u_2}(A_1 h_1)\right]\vec{e_3}$$

まとめは左ページへ