Ⅰ．マクスウェルの電磁方程式のテンソル表示への書き換え

電気量保存の式より

$$\frac{\partial \rho}{\partial t} + div\vec{j} = 0 \cdots ①$$

４元ベクトルの形式に書き換えると

$$Div\vec{j} = \frac{\partial (c\rho)}{\partial t} + \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} = \frac{\partial (c\rho)}{\partial x^0} + \frac{\partial j^1}{\partial x^1} + \frac{\partial j^2}{\partial x^2} + \frac{\partial j^3}{\partial x^3} = 0$$

$$Div\vec{j} = \frac{\partial j^0}{\partial x^0} + \frac{\partial j^1}{\partial x^1} + \frac{\partial j^2}{\partial x^2} + \frac{\partial j^3}{\partial x^3} = 0 \cdots ②$$

$\vec{j} = \begin{bmatrix} c\rho & j_x & j_y & j_z \end{bmatrix} = j^\mu$ とおけば

$$\partial_\mu j^\mu = 0 \cdots ③$$

$$div\vec{D} = \rho \cdots (3) \qquad rot\vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j} \cdots (2)$$

$$div\vec{B} = 0 \cdots (1) \qquad rot\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \cdots (4)$$

$ct = x^0, x = x^1, y = x^2, z = x^3$　とおいて、(2)(3)を書き換えると

$$0 \quad + c\frac{\partial D_x}{\partial x^1} \quad + c\frac{\partial D_y}{\partial x^2} \quad + c\frac{\partial D_z}{\partial x^3} = c\rho \cdots ⑤$$

$$-c\frac{\partial D_x}{\partial x^0} \quad + 0 \quad + \frac{\partial H_z}{\partial x^2} \quad - \frac{\partial H_y}{\partial x^3} = j_x \cdots ⑥$$

$$-c\frac{\partial D_y}{\partial x^0} \quad - \frac{\partial H_z}{\partial x^1} \quad + 0 \quad + \frac{\partial H_x}{\partial x^3} = j_y \cdots ⑦$$

$$-c\frac{\partial D_z}{\partial x^0} \quad + \frac{\partial H_y}{\partial x^1} \quad - \frac{\partial H_x}{\partial x^2} \quad + 0 = j_z \cdots ⑧ \qquad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \qquad c^2 = \frac{1}{\varepsilon_0 \mu_0}$$

$$0 \quad + c\frac{\partial \varepsilon_0 \mu_0 E_x}{\partial x^1} \quad + c\frac{\partial \varepsilon_0 \mu_0 E_y}{\partial x^2} \quad + c\frac{\partial \varepsilon_0 \mu_0 E_z}{\partial x^3} = \mu_0 c\rho \cdots ⑤$$

$$-c\frac{\partial \varepsilon_0 \mu_0 E_x}{\partial x^0} \quad + 0 \quad + \frac{\partial \mu_0 H_z}{\partial x^2} \quad - \frac{\partial \mu_0 H_y}{\partial x^3} = \mu_0 j_x \cdots ⑥$$

$$-c\frac{\partial \varepsilon_0 \mu_0 E_y}{\partial x^0} \quad \frac{\partial \mu_0 H_z}{\partial x^1} \quad + 0 \quad - \frac{\partial \mu_0 H_x}{\partial x^3} = \mu_0 j_y \cdots ⑦$$

$$-c\frac{\partial \varepsilon_0 \mu_0 E_z}{\partial x^0} \quad + \frac{\partial \mu_0 H_y}{\partial x^1} \quad - \frac{\partial \mu_0 H_x}{\partial x^2} \quad + 0 = \mu_0 j_z \cdots ⑧$$

$$0 \quad +\frac{\partial}{\partial x^1}\left(\frac{E_x}{c}\right) \quad +\frac{\partial}{\partial x^2}\left(\frac{E_y}{c}\right) \quad +\frac{\partial}{\partial x^3}\left(\frac{E_z}{c}\right) = \mu_0 c\rho \cdots ⑨$$

$$\frac{\partial}{\partial x^0}\left(-\frac{E_x}{c}\right) \quad +0 \quad +\frac{\partial(B_z)}{\partial x^2} \quad +\frac{\partial(-B_y)}{\partial x^3} = \mu_0 j_x \cdots ⑩$$

$$\frac{\partial}{\partial x^0}\left(-\frac{E_y}{c}\right) \quad +\frac{\partial(-B_z)}{\partial x^1} \quad +0 \quad +\frac{\partial B_x}{\partial x^3} = \mu_0 j_y \cdots ⑪$$

$$\frac{\partial}{\partial x^0}\left(-\frac{E_z}{c}\right) \quad +\frac{\partial B_y}{\partial x^1} \quad +\frac{\partial(-B_x)}{\partial x^2} \quad +0 = \mu_0 j_z \cdots ⑫$$

$$\begin{bmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{bmatrix} = \left(F^{\mu\nu}\right) \cdots ⑬$$ とおくと⑨～⑫は $\frac{\partial}{\partial x^\mu} F^{\mu\nu} = \mu_0 j^\nu$ $\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \cdots ⑭$

次に(1)(4)を書き換えると

$$div\vec{B} = 0 \qquad rot\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$0 \quad -\frac{\partial B_x}{\partial x^1} \quad -\frac{\partial B_y}{\partial x^2} \quad -\frac{\partial B_z}{\partial x^3} = 0 \cdots ⑮$$

$$\frac{\partial B_x}{\partial x^0} \quad +0 \quad +\frac{\partial}{\partial x^2}\left(\frac{E_z}{c}\right) \quad -\frac{\partial}{\partial x^3}\left(\frac{E_y}{c}\right) = 0 \cdots ⑯$$

$$\frac{\partial B_y}{\partial x^0} \quad -\frac{\partial}{\partial x^1}\left(\frac{E_z}{c}\right) \quad +0 \quad +\frac{\partial}{\partial x^3}\left(\frac{E_x}{c}\right) = 0 \cdots ⑰$$

$$\frac{\partial B_z}{\partial x^0} \quad +\frac{\partial}{\partial x^1}\left(\frac{E_y}{c}\right) \quad -\frac{\partial}{\partial x^2}\left(\frac{E_x}{c}\right) \quad +0 = 0 \cdots ⑱$$

$$\begin{bmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{bmatrix} = \left(F^{\mu\nu}\right) \cdots ⑲$$ $\frac{\partial F^{jk}}{\partial x_i} + \frac{\partial F^{ij}}{\partial x_k} + \frac{\partial F^{ki}}{\partial x_j} = 0$

$ct = x_0, -x = x_1, -y = x_2, -z = x_3$

(ｱ) $i=1 \quad j=2 \quad k=3$ のとき $\rightarrow \dfrac{\partial F^{23}}{\partial x_1} + \dfrac{\partial F^{12}}{\partial x_3} + \dfrac{\partial F^{31}}{\partial x_2} = 0 \rightarrow \dfrac{\partial(-B_x)}{-\partial x} + \dfrac{\partial(-B_z)}{-\partial z} + \dfrac{\partial(-B_y)}{-\partial y} = 0 \rightarrow \ div\vec{B} = 0$

(ｲ) $i=0 \quad j=1 \quad k=2$ のとき

$\dfrac{\partial F^{12}}{\partial x_0} + \dfrac{\partial F^{01}}{\partial x_2} + \dfrac{\partial F^{20}}{\partial x_1} = 0$ は次の通り

$\dfrac{\partial(-B_z)}{\partial x_0} + \dfrac{\partial}{\partial x_2}\left(-\dfrac{E_x}{c}\right) + \dfrac{\partial}{\partial x_1}\left(\dfrac{E_y}{c}\right) = 0 \quad \dfrac{1}{c}\dfrac{\partial(-B_z)}{\partial t} + \dfrac{\partial}{(-\partial y)}\left(-\dfrac{E_x}{c}\right) + \dfrac{\partial}{(-\partial x)}\left(\dfrac{E_y}{c}\right) = 0 \quad -\dfrac{1}{c}\dfrac{\partial B_z}{\partial t} - \dfrac{1}{c}\left(\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y}\right) = 0$

$-\dfrac{1}{c}\left(rot\vec{E} + \dfrac{\partial \vec{B}}{\partial t}\right)_z = 0 \quad \rightarrow \quad \left(rot\vec{E} + \dfrac{\partial \vec{B}}{\partial t}\right)_z = 0$

以下サイクリックに取り扱えば

(ｳ) $i=0 \quad j=2 \quad k=3$ のとき

$\left(rot\vec{E} + \dfrac{\partial \vec{B}}{\partial t}\right)_x = 0$

(ｴ) $i=0 \quad j=3 \quad k=1$ のとき

$\left(rot\vec{E} + \dfrac{\partial \vec{B}}{\partial t}\right)_y = 0$

$i=0$ の場合

$\dfrac{\partial F^{jk}}{\partial x_i} + \dfrac{\partial F^{ij}}{\partial x_k} + \dfrac{\partial F^{ki}}{\partial x_j} = 0 \quad \rightarrow \quad rot\vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0$

まとめると　４つ方程式が二本の式で表現できる。

（マックスウェル電磁方程式のテンソル表示）

$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \quad \Leftrightarrow \quad div\vec{D} = \rho \quad rot\vec{H} - \dfrac{\partial \vec{D}}{\partial t} = \vec{j}$

$\partial^\mu F^{\nu\lambda} + \partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} = 0 \quad \Leftrightarrow \quad div\vec{B} = 0 \quad rot\vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0$

$A^{\mu}=\left[\frac{\phi}{c}\quad A_x\quad A_y\quad A_z\right]$ とおくと

$$F^{\mu\nu}=\frac{\partial A^{\nu}}{\partial x_{\mu}}-\frac{\partial A^{\mu}}{\partial x_{\nu}}=\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}$$

$$F^{00}=\frac{\partial A^0}{\partial x_0}-\frac{\partial A^0}{\partial x_0}=0\quad F^{01}=\frac{\partial A^1}{\partial x_0}-\frac{\partial A^0}{\partial x_1}=\frac{\partial A_x}{c\partial t}+\frac{\partial \phi}{c\partial x}=-\frac{E_x}{c}\quad F^{02}=\frac{\partial A^2}{\partial x_0}-\frac{\partial A^0}{\partial x_2}=\frac{\partial A_y}{c\partial t}+\frac{\partial \phi}{c\partial y}=-\frac{E_y}{c}\quad F^{03}=\frac{\partial A^3}{\partial x_0}-\frac{\partial A^0}{\partial x_3}=\frac{\partial A_z}{c\partial t}+\frac{\partial \phi}{c\partial z}=-\frac{E_z}{c}$$

$$F^{10}=\frac{\partial A^0}{\partial x_1}-\frac{\partial A^1}{\partial x_0}=-\frac{\partial \phi}{c\partial x}-\frac{\partial A_x}{c\partial t}=\frac{E_x}{c}\quad F^{11}=\frac{\partial A^1}{\partial x_1}-\frac{\partial A^1}{\partial x_1}=0\quad F^{12}=\frac{\partial A^2}{\partial x_1}-\frac{\partial A^1}{\partial x_2}=-\frac{\partial A_y}{\partial x}+\frac{\partial A_x}{\partial y}=-B_z\quad F^{13}=\frac{\partial A^3}{\partial x_1}-\frac{\partial A^1}{\partial x_3}=-\frac{\partial A_z}{\partial x}+\frac{\partial A_x}{\partial z}=B_y$$

$$F^{20}=\frac{\partial A^0}{\partial x_2}-\frac{\partial A^2}{\partial x_0}=-\frac{\partial \phi}{c\partial y}-\frac{\partial A_y}{c\partial t}=\frac{E_y}{c}\quad F^{21}=\frac{\partial A^1}{\partial x_2}-\frac{\partial A^2}{\partial x_1}=-\frac{\partial A_x}{\partial y}+\frac{\partial A_y}{\partial x}=B_z\quad F^{22}=\frac{\partial A^2}{\partial x_2}-\frac{\partial A^2}{\partial x_2}=0\quad F^{23}=\frac{\partial A^3}{\partial x_2}-\frac{\partial A^2}{\partial x_3}=-\frac{\partial A_z}{\partial y}+\frac{\partial A_y}{\partial z}=-B_x$$

$$F^{30}=\frac{\partial A^0}{\partial x_3}-\frac{\partial A^3}{\partial x_0}=-\frac{\partial \phi}{c\partial z}-\frac{\partial A_z}{c\partial t}=\frac{E_z}{c}\quad F^{31}=\frac{\partial A^1}{\partial x_3}-\frac{\partial A^3}{\partial x_1}=-\frac{\partial A_x}{\partial z}+\frac{\partial A_z}{\partial x}=-B_y\quad F^{32}=\frac{\partial A^2}{\partial x_3}-\frac{\partial A^3}{\partial x_2}=\frac{\partial A_y}{\partial z}-\frac{\partial A_z}{\partial y}=B_x\quad F^{33}=\frac{\partial A^3}{\partial x_3}-\frac{\partial A^3}{\partial x_3}=0$$

$$\begin{bmatrix}0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c}\\ \frac{E_x}{c} & 0 & -B_z & B_y\\ \frac{E_y}{c} & B_z & 0 & -B_x\\ \frac{E_z}{c} & -B_y & B_x & 0\end{bmatrix}=\left(F^{\mu\nu}\right)$$

$$F_{\mu\nu}=\frac{\partial A_{\nu}}{\partial x^{\mu}}-\frac{\partial A_{\mu}}{\partial x^{\nu}}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}$$

$$F_{00}=\frac{\partial A_0}{\partial x^0}-\frac{\partial A_0}{\partial x^0}=0\quad F_{01}=\frac{\partial A_1}{\partial x^0}-\frac{\partial A_0}{\partial x^1}=-\frac{\partial A_x}{c\partial t}-\frac{\partial \phi}{c\partial x}=\frac{E_x}{c}\quad F_{02}=\frac{\partial A_2}{\partial x^0}-\frac{\partial A_0}{\partial x^2}=-\frac{\partial A_y}{c\partial t}-\frac{\partial \phi}{c\partial y}=\frac{E_y}{c}\quad F_{03}=\frac{\partial A_3}{\partial x^0}-\frac{\partial A_0}{\partial x^3}=-\frac{\partial A_z}{c\partial t}-\frac{\partial \phi}{c\partial z}=\frac{E_z}{c}$$

$$F_{10}=\frac{\partial A_0}{\partial x^1}-\frac{\partial A_1}{\partial x^0}=\frac{\partial \phi}{c\partial x}+\frac{\partial A_x}{c\partial t}=-\frac{E_x}{c}\quad F_{11}=\frac{\partial A_1}{\partial x^1}-\frac{\partial A_1}{\partial x^1}=0\quad F_{12}=\frac{\partial A_2}{\partial x^1}-\frac{\partial A_1}{\partial x^2}=-\frac{\partial A_y}{\partial x}+\frac{\partial A_x}{\partial y}=-B_z\quad F_{13}=\frac{\partial A_3}{\partial x^1}-\frac{\partial A_1}{\partial x^3}=-\frac{\partial A_z}{\partial x}+\frac{\partial A_x}{\partial z}=B_y$$

$$F_{20}=\frac{\partial A_0}{\partial x^2}-\frac{\partial A_2}{\partial x^0}=\frac{\partial \phi}{c\partial y}+\frac{\partial A_y}{c\partial t}=-\frac{E_y}{c}\quad F_{21}=\frac{\partial A_1}{\partial x^2}-\frac{\partial A_2}{\partial x^1}=-\frac{\partial A_x}{\partial y}+\frac{\partial A_y}{\partial x}=B_z\quad F_{22}=\frac{\partial A_2}{\partial x^2}-\frac{\partial A_2}{\partial x^2}=0\quad F_{23}=\frac{\partial A_3}{\partial x^2}-\frac{\partial A_2}{\partial x^3}=-\frac{\partial A_z}{\partial y}+\frac{\partial A_y}{\partial z}=-B_x$$

$$F_{30}=\frac{\partial A_0}{\partial x^3}-\frac{\partial A_3}{\partial x^0}=-\frac{\partial \phi}{c\partial z}-\frac{\partial A_z}{c\partial t}=\frac{E_z}{c}\quad F_{31}=\frac{\partial A_1}{\partial x^3}-\frac{\partial A_3}{\partial x^1}=-\frac{\partial A_x}{\partial z}+\frac{\partial A_z}{\partial x}=-B_y\quad F_{32}=\frac{\partial A_2}{\partial x^3}-\frac{\partial A_3}{\partial x^2}=-\frac{\partial A_y}{\partial z}+\frac{\partial A_z}{\partial y}=B_x\quad F_{33}=\frac{\partial A_3}{\partial x^3}-\frac{\partial A_3}{\partial x^3}=0$$

$$\begin{bmatrix}0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c}\\ -\frac{E_x}{c} & 0 & -B_z & B_y\\ -\frac{E_y}{c} & B_z & 0 & -B_x\\ -\frac{E_z}{c} & -B_y & B_x & 0\end{bmatrix}=\left(F_{\mu\nu}\right)$$