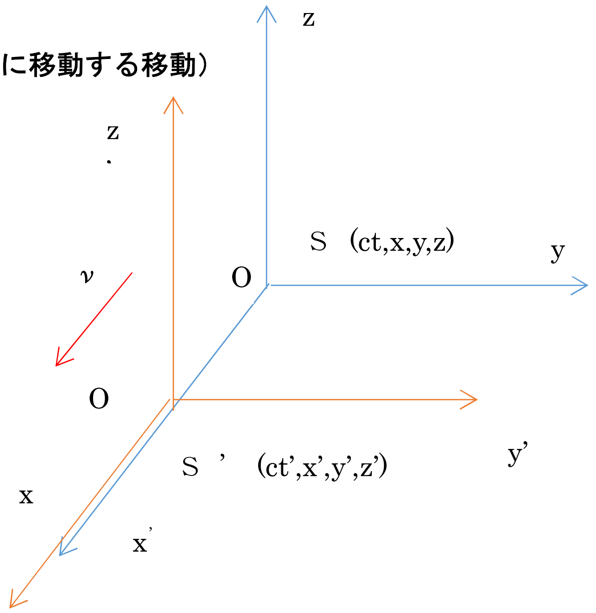
ローレンツ力の導出過程

Ⅰ．数学的な準備（Ｓ’系はＳ系とｘ軸を重ねた状態で正の向きに移動する移動）



$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \quad \cdots\cdot ①$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma ct - \gamma\beta x \\ -\gamma\beta ct + \gamma x \\ y \\ z \end{bmatrix} \quad \cdots\cdot ②$$

$$ct' = \gamma ct - \gamma\beta x \qquad\qquad x' = -\gamma\beta ct + \gamma x$$

$$\frac{\partial ct'}{\partial ct} = \gamma \quad \cdots\cdot ③ \qquad\qquad \frac{\partial x'}{\partial ct} = -\gamma\beta \quad \cdots\cdot ④$$

$$\frac{\partial ct'}{\partial x} = -\gamma\beta \qquad\qquad \frac{\partial x'}{\partial x} = \gamma$$

$[ct \quad x \quad y \quad z] = [x^0 \quad x^1 \quad x^2 \quad x^3]$ の時空間の変数でまずは取り扱う。

$$\frac{\partial f(ct',x')}{\partial x} = \frac{\partial f}{\partial ct'}\frac{\partial ct'}{\partial x} + \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} = -\gamma\beta\frac{\partial f}{\partial ct'} + \gamma\frac{\partial f}{\partial x'} = \gamma\left(-\beta\frac{\partial}{\partial x^{0}{}'} + \frac{\partial}{\partial x^{1}{}'}\right)f \quad \cdots\cdot ⑤$$

$$\frac{\partial f(ct',x')}{\partial ct} = \frac{\partial f}{\partial ct'}\frac{\partial ct'}{\partial ct} + \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial ct} = \gamma\frac{\partial f}{\partial ct'} - \gamma\beta\frac{\partial f}{\partial x'} = \gamma\left(\frac{\partial}{\partial x^{0}{}'} - \beta\frac{\partial}{\partial x^{1}{}'}\right)f \quad \cdots\cdot ⑥$$

⑥より $$\frac{\partial f(ct',x')}{\partial x^0} = \gamma\left(\frac{\partial}{\partial x^{0}{}'} - \beta\frac{\partial}{\partial x^{1}{}'}\right)f \quad \cdots\cdot ⑦$$

⑤より $$\frac{\partial f(ct',x')}{\partial x^1} = \gamma\left(-\beta\frac{\partial}{\partial x^{0}{}'} + \frac{\partial}{\partial x^{1}{}'}\right)f \quad \cdots\cdot ⑧$$

β×⑦＋⑧より $$\beta\frac{\partial f(ct',x')}{\partial x^0} = \left(\gamma\beta\frac{\partial}{\partial x^{0}{}'} - \gamma\beta^2\frac{\partial}{\partial x^{1}{}'}\right)f \quad \cdots\cdot ⑨ \qquad \frac{\partial f(ct',x')}{\partial x^1} = \left(-\gamma\beta\frac{\partial}{\partial x^{0}{}'} + \gamma\frac{\partial}{\partial x^{1}{}'}\right)f \quad \cdots\cdot ⑧$$

⑨＋⑧より $$\beta\frac{\partial f(ct',x')}{\partial x^0} + \frac{\partial f(ct',x')}{\partial x^1} = \gamma\left(1-\beta^2\right)\frac{\partial f}{\partial x^{1}{}'} = \frac{1}{\gamma}\frac{\partial f}{\partial x^{1}{}'} \quad \cdots\cdot ⑩$$

$$\frac{\partial f}{\partial x^{1}{}'} = \gamma\left(\beta\frac{\partial f(ct',x')}{\partial x^0} + \frac{\partial f(ct',x')}{\partial x^1}\right) \quad \cdots\cdot ⑪$$

⑦＋β×⑧より $\dfrac{\partial f(ct',x')}{\partial x^0}+\beta\dfrac{\partial f(ct',x')}{\partial x^1}=\gamma\left(1-\beta^2\right)\dfrac{\partial f}{\partial x^{0}{}'}=\dfrac{1}{\gamma}\dfrac{\partial f}{\partial x^{0}{}'}$ ････⑫

$\dfrac{\partial f}{\partial x^{0}{}'}=\gamma\left(\dfrac{\partial f(ct',x')}{\partial x^0}+\beta\dfrac{\partial f(ct',x')}{\partial x^1}\right)$ ････⑬

（A）時空間変数での扱い

$$\frac{\partial}{\partial x^{0}{}'}=\gamma\left(\frac{\partial f(ct',x')}{\partial x^0}+\beta\frac{\partial f(ct',x')}{\partial x^1}\right)\qquad \frac{\partial f(ct',x')}{\partial x^0}=\gamma\left(\frac{\partial}{\partial x^{0}{}'}-\beta\frac{\partial}{\partial x^{1}{}'}\right)f$$

$$\frac{\partial f}{\partial x^{1}{}'}=\gamma\left(\beta\frac{\partial f(ct',x')}{\partial x^0}+\frac{\partial f(ct',x')}{\partial x^1}\right)\qquad \frac{\partial f(ct',x')}{\partial x^1}=\gamma\left(-\beta\frac{\partial}{\partial x^{0}{}'}+\frac{\partial}{\partial x^{1}{}'}\right)f$$

（B）時空間変数での扱い

$$\frac{\partial}{c\partial t'}=\gamma\left(\frac{1}{c}\frac{\partial f(ct',x')}{\partial t}+\beta\frac{\partial f(ct',x')}{\partial x}\right)\qquad \frac{\partial f(ct',x')}{c\partial t}=\gamma\left(\frac{1}{c}\frac{\partial}{\partial t'}-\beta\frac{\partial}{\partial x'}\right)f\qquad \frac{\partial f(ct',x')}{\partial z}=\frac{\partial f(ct',x')}{\partial z'}$$

$$\frac{\partial f}{\partial x'}=\gamma\left(\frac{\beta}{c}\frac{\partial f(ct',x')}{\partial t}+\frac{\partial f(ct',x')}{\partial x}\right)\qquad \frac{\partial f(ct',x')}{\partial x}=\gamma\left(-\frac{\beta}{c}\frac{\partial}{\partial t'}+\frac{\partial}{\partial x'}\right)f\qquad \frac{\partial f(ct',x')}{\partial y}=\frac{\partial f(ct',x')}{\partial y'}$$

（C）$[t\quad x\quad y\quad z]$ での扱い

$$\frac{\partial}{\partial t'}=\gamma\left(\frac{\partial f(ct',x')}{\partial t}+v\frac{\partial f(ct',x')}{\partial x}\right)\qquad \frac{\partial f(ct',x')}{\partial t}=\gamma\left(\frac{\partial}{\partial t'}-v\frac{\partial}{\partial x'}\right)f\qquad \frac{\partial f(ct',x')}{\partial z}=\frac{\partial f(ct',x')}{\partial z'}$$

$$\frac{\partial f}{\partial x'}=\gamma\left(\frac{v}{c^2}\frac{\partial f(ct',x')}{\partial t}+\frac{\partial f(ct',x')}{\partial x}\right)\qquad \frac{\partial f(ct',x')}{\partial x}=\gamma\left(-\frac{v}{c^2}\frac{\partial}{\partial t'}+\frac{\partial}{\partial x'}\right)f\qquad \frac{\partial f(ct',x')}{\partial y}=\frac{\partial f(ct',x')}{\partial y'}$$

Ⅱ．マクスウェル方程式のローレンツ変換による書き換え

$rot\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}\qquad div\vec{B}=0$　　まずこの二つの方程式を元にして書き換えをする。

（1）x成分について

$\dfrac{\partial E_z}{\partial y}-\dfrac{\partial E_y}{\partial z}=-\dfrac{\partial B_x}{\partial t}$ ････①　$\dfrac{\partial E_z}{\partial y'}-\dfrac{\partial E_y}{\partial z'}=-\dfrac{\partial B_x}{\partial t}$ ････②

$\dfrac{\partial E_z}{\partial y'}-\dfrac{\partial E_y}{\partial z'}=-\gamma\left(\dfrac{\partial}{\partial t'}-v\dfrac{\partial}{\partial x'}\right)B_x=-\gamma\dfrac{\partial B_x}{\partial t'}+\gamma v\dfrac{\partial B_x}{\partial x'}$ ････③

$div\vec{B}=0$ より $\dfrac{\partial B_x}{\partial x}+\dfrac{\partial B_y}{\partial y}+\dfrac{\partial B_z}{\partial z}=0$　　$\gamma\left(-\dfrac{v}{c^2}\dfrac{\partial}{\partial t'}+\dfrac{\partial}{\partial x'}\right)B_x+\dfrac{\partial B_y}{\partial y'}+\dfrac{\partial B_z}{\partial z'}=0$ ････④

$-\gamma\dfrac{v}{c^2}\dfrac{\partial B_x}{\partial t'}+\gamma\dfrac{\partial B_x}{\partial x'}+\dfrac{\partial B_y}{\partial y'}+\dfrac{\partial B_z}{\partial z'}=0$　　$\gamma\dfrac{\partial B_x}{\partial x'}=-\left(\dfrac{\partial B_y}{\partial y'}+\dfrac{\partial B_z}{\partial z'}\right)+\gamma\dfrac{v}{c^2}\dfrac{\partial B_x}{\partial t'}$ ････⑤

$$\gamma v\frac{\partial B_x}{\partial x'}=-v\left(\frac{\partial B_y}{\partial y'}+\frac{\partial B_z}{\partial z'}\right)+\gamma\frac{v^2}{c^2}\frac{\partial B_x}{\partial t'}$$ ････⑥を③の右辺に代入すると

$$\frac{\partial E_z}{\partial y'}-\frac{\partial E_y}{\partial z'}=-\gamma\frac{\partial B_x}{\partial t'}+\gamma v\frac{\partial B_x}{\partial x'}=-\gamma\frac{\partial B_x}{\partial t'}+\gamma\frac{v^2}{c^2}\frac{\partial B_x}{\partial t'}-v\left(\frac{\partial B_y}{\partial y'}+\frac{\partial B_z}{\partial z'}\right)\cdots\cdot⑦$$

$$\frac{\partial E_z}{\partial y'}-\frac{\partial E_y}{\partial z'}=-\gamma\left(1-\beta^2\right)\frac{\partial B_x}{\partial t'}-v\left(\frac{\partial B_y}{\partial y'}+\frac{\partial B_z}{\partial z'}\right)\qquad\frac{\partial E_z}{\partial y'}-\frac{\partial E_y}{\partial z'}=-\frac{1}{\gamma}\frac{\partial B_x}{\partial t'}-v\left(\frac{\partial B_y}{\partial y'}+\frac{\partial B_z}{\partial z'}\right)$$

$$\frac{\partial E_z}{\partial y'}-\frac{\partial E_y}{\partial z'}+v\left(\frac{\partial B_y}{\partial y'}+\frac{\partial B_z}{\partial z'}\right)=-\frac{1}{\gamma}\frac{\partial B_x}{\partial t'}\qquad\frac{\partial}{\partial y'}\left(E_z+vB_y\right)-\frac{\partial}{\partial z'}\left(E_y-vB_z\right)=-\frac{1}{\gamma}\frac{\partial B_x}{\partial t'}\cdots\cdot⑨$$

$$\frac{\partial}{\partial y'}\gamma\left(E_z+vB_y\right)-\frac{\partial}{\partial z'}\gamma\left(E_y-vB_z\right)=-\frac{\partial B_x}{\partial t'}\cdots\cdot⑩$$

$$\left(\vec{E}'\right)_z=\gamma\left(E_z+vB_y\right)=\gamma\left(\vec{E}+\vec{v}\times\vec{B}\right)_z$$
$$\left(\vec{E}'\right)_y=\gamma\left(E_y-vB_z\right)=\gamma\left(\vec{E}+\vec{v}\times\vec{B}\right)_y$$
$$\left(\vec{B}'\right)_x=\left(\vec{B}\right)_x$$ ････⑪ とすると

変換後の電場は $\vec{E}'=\gamma\left(\vec{E}+\vec{v}\times\vec{B}\right)=\dfrac{\vec{E}+\vec{v}\times\vec{B}}{\sqrt{1-\beta^2}}$ と変換されると予測できる。

（2） y成分について

$$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-\frac{\partial B_y}{\partial t}\cdots\cdot⑫\qquad\frac{\partial E_x}{\partial z'}-\gamma\left(-\frac{v}{c^2}\frac{\partial}{\partial t'}+\frac{\partial}{\partial x'}\right)E_z=-\gamma\left(\frac{\partial}{\partial t'}-v\frac{\partial}{\partial x'}\right)B_y\cdots\cdot⑬$$

$$\frac{\partial E_x}{\partial z'}-\gamma\frac{\partial E_z}{\partial x'}-\gamma v\frac{\partial B_y}{\partial x'}=-\gamma\frac{\partial B_y}{\partial t'}-\gamma\frac{v}{c^2}\frac{\partial E_z}{\partial t'}\cdots\cdot⑭$$

$$\frac{\partial E_x}{\partial z'}-\frac{\partial\gamma\left(E_z+vB_y\right)}{\partial x'}=-\gamma\frac{\partial B_y}{\partial t'}-\gamma\frac{v}{c^2}\frac{\partial E_z}{\partial t'}=-\frac{\partial}{\partial t'}\gamma\left(B_y+\frac{v}{c^2}E_z\right)$$

$$\frac{\partial E_x}{\partial z'}-\frac{\partial\gamma\left(E_z+vB_y\right)}{\partial x'}=-\frac{\partial}{\partial t'}\gamma\left(B_y+\frac{v}{c^2}E_z\right)\cdots\cdot⑮$$

$$\left(\vec{E}'\right)_x=\left(\vec{E}\right)_x\qquad\left(\vec{E}'\right)_z=\gamma\left(E_z+vB_y\right)=\gamma\left(\vec{E}+\vec{v}\times\vec{B}\right)_z\qquad\left(\vec{B}'\right)_y=\gamma\left(B_y+\frac{v}{c^2}E_z\right)=\gamma\left(\vec{B}-\frac{\vec{v}}{c^2}\times\vec{E}\right)_y\cdots\cdot⑯$$

$\vec{B}'=\gamma\left(\vec{B}-\dfrac{\vec{v}}{c^2}\times\vec{E}\right)$ と変換されることが予測できる。

（3） z成分について

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-\frac{\partial B_z}{\partial t}\cdots\cdot⑰\qquad\gamma\left(-\frac{v}{c^2}\frac{\partial}{\partial t'}+\frac{\partial}{\partial x'}\right)E_y-\frac{\partial E_x}{\partial y}=-\gamma\left(\frac{\partial}{\partial t'}-v\frac{\partial}{\partial x'}\right)B_z\cdots\cdot⑱$$

$$-\gamma\frac{v}{c^2}\frac{\partial E_y}{\partial t'}+\gamma\frac{\partial E_y}{\partial x'}-\frac{\partial E_x}{\partial y}=-\gamma\frac{\partial B_z}{\partial t'}+\gamma v\frac{\partial B_z}{\partial x'}\qquad+\gamma\frac{\partial E_y}{\partial x'}-\gamma v\frac{\partial B_z}{\partial x'}-\frac{\partial E_x}{\partial y}=-\gamma\frac{\partial B_z}{\partial t'}+\gamma\frac{v}{c^2}\frac{\partial E_y}{\partial t'}$$

$$\frac{\partial}{\partial x'}\gamma\left(E_y - vB_z\right) - \frac{\partial E_x}{\partial y'} = -\frac{\partial}{\partial t'}\gamma\left(B_z - \frac{v}{c^2}E_y\right) \cdots\cdot ⑲$$

$$\left(\vec{E}'\right)_y = \gamma\left(E_y - vB_z\right) = \gamma\left(\vec{E} + \vec{v}\times\vec{B}\right)_y$$

$$\left(\vec{B}'\right)_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right) = \gamma\left(\vec{B} - \frac{\vec{v}}{c^2}\times\vec{E}\right)_z \qquad \left(\vec{E}'\right)_x = \left(\vec{E}\right)_x \quad \cdots\cdot ⑳$$

（まとめ）

ローレンツ変換に従って、マクスウェルの方程式を書き換えた場合に

電場 $\vec{E}$ と磁場 $\vec{B}$ が次の通りに変換されるなら $rot\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad div\vec{B} = 0$ はローレンツ変換において

対称性（共変性）を保つ。

電場 $\vec{E}$ と磁場 $\vec{B}$ のローレンツ変換における変換式

ｘ軸方向（Ｓ’系の原点Ｏ’の進行方向）について

$$\left(\vec{E}'\right)_x = \left(\vec{E}\right)_x \qquad \left(\vec{B}'\right)_x = \left(\vec{B}\right)_x$$

ｙ、ｚ軸方向（Ｓ’系の原点Ｏ’の進行方向に垂直な方向）について

$$\vec{E}' = \gamma\left(\vec{E} + \vec{v}\times\vec{B}\right) = \frac{\vec{E} + \vec{v}\times\vec{B}}{\sqrt{1-\beta^2}} \qquad \vec{B}' = \gamma\left(\vec{B} - \frac{\vec{v}}{c^2}\times\vec{E}\right) = \frac{\vec{B} - \frac{\vec{v}}{c^2}\times\vec{E}}{\sqrt{1-\beta^2}}$$