(相対論的力学)

$$(x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

$$\begin{cases} ct' = \gamma ct - \gamma\beta x = \gamma(ct - \beta x) \\ x' = -\beta\gamma ct + \gamma x = \gamma(x - \ \ t) \\ y' = y \\ z' = z \end{cases} \quad \text{—①} \qquad \begin{cases} \gamma = \dfrac{1}{\sqrt{1-\beta^2}} \\ \dfrac{v}{c} = \beta \end{cases}$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \quad \text{—②}$$

vは K系に対する K'系の速度　x軸の正の向きに移動

$\beta = \dfrac{v}{c}$

$$s'^2 = (ct')^2 - x'^2 - y'^2 - z'^2$$
$$= \gamma^2(ct - \beta x)^2 - \gamma^2(x - vt)^2 - y^2 - z^2$$
$$\gamma^2\{c^2t^2 - 2c\beta xt + \beta^2x^2 - x^2 + 2xvt - v^2t^2\} - y^2 - z^2$$
$$\gamma^2\{(c^2 - v^2)t^2 - (1-\beta^2)x^2\} - y^2 - z^2$$
$$\gamma^2 c^2(1-\beta^2)t^2 - \gamma^2(1-\beta^2)x^2 - y^2 - z^2$$
$$= c^2t^2 - x^2 - y^2 - z^2 \quad \text{—③}$$

$$dx^\mu dx_\mu = dx^\mu g_{\mu\nu} dx^\nu$$

$\therefore$ $s'^2 = s^2 = (ct)^2 - x^2 - y^2 - z^2 = (\text{一定})$ —④ Lorentz scalar

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = dx^\mu dx_\mu$$
$$c^2dt^2 - dx^2 - dy^2 - dz^2$$
$$(c^2 - v^2)dt^2 \quad \text{—⑤}$$

ここからは
($v^2 = v_x^2 + v_y^2 + v_z^2$ を意味する)

vは物体の速度 K系の速度を示すのとは異なる

粒子の S系上での速度の2乗を意味する

$$ds = \sqrt{c^2 - v^2}\,dt$$
$$= c\sqrt{1 - \frac{v^2}{c^2}}\,dt$$

$$\boxed{d\tau = \frac{ds}{c} = \sqrt{1 - \frac{v^2}{c^2}}\,dt \quad (\text{固有時}) \quad \text{—⑥}}$$

四元速度は4元ベクトル

$$\frac{dx^\mu}{d\tau} = \left(\frac{c\,dt}{\sqrt{1-\beta^2}dt}, \frac{dx}{\sqrt{1-\beta^2}dt}, \frac{dy}{\sqrt{1-\beta^2}dt}, \frac{dz}{\sqrt{1-\beta^2}dt}\right) \quad \text{—⑦}$$

$$\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{\sqrt{1-\beta^2}dt} = \frac{1}{\sqrt{1-\beta^2}}\frac{dx^\mu}{dt} = \left(\frac{c}{\sqrt{1-\beta^2}}, \frac{1}{\sqrt{1-\beta^2}}\frac{dx}{dt}, \frac{1}{\sqrt{1-\beta^2}}\frac{dy}{dt}, \frac{1}{\sqrt{1-\beta^2}}\frac{dz}{dt}\right)$$
$$= \left(\frac{c}{\sqrt{1-\beta^2}}, \frac{v_x}{\sqrt{1-\beta^2}}, \frac{v_y}{\sqrt{1-\beta^2}}, \frac{v_z}{\sqrt{1-\beta^2}}\right) \quad \text{—⑧}$$

更に 4元運動量も四元ベクトル

$$p^\mu = \frac{m_0}{\sqrt{1-\beta^2}}\frac{dx^\mu}{dt} = \left(\frac{m_0c}{\sqrt{1-\beta^2}}, \frac{m_0v_x}{\sqrt{1-\beta^2}}, \frac{m_0v_y}{\sqrt{1-\beta^2}}, \frac{m_0v_z}{\sqrt{1-\beta^2}}\right) \quad \text{—⑨}$$

更に

$$\frac{dp^\mu}{d\tau} = \frac{1}{\sqrt{1-\beta^2}}\frac{dp^\mu}{dt} \quad \text{—⑩}$$
$$= \frac{1}{\sqrt{1-\beta^2}}\frac{d}{dt}\left(\frac{m_0c}{\sqrt{1-\beta^2}}, \frac{m_0v_x}{\sqrt{1-\beta^2}}, \frac{m_0v_y}{\sqrt{1-\beta^2}}, \frac{m_0v_z}{\sqrt{1-\beta^2}}\right) \quad \text{—⑪}$$

$m = \dfrac{m_0}{\sqrt{1-\beta^2}}$ とおく

$$\left.\begin{aligned} \frac{d}{dt}(mv_x) &= F_x \\ \frac{d}{dt}(mv_y) &= F_y \\ \frac{d}{dt}(mv_z) &= F_z \end{aligned}\right\} \text{とおくと}$$

物体が x軸方向に一定の力 F を受けて運動すると

$$\frac{d}{dt}\left(\frac{m_0}{\sqrt{1-\beta^2}}\frac{dx}{dt}\right) = F$$
$$m_0\frac{d}{dt}\left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{dx}{dt}\right] = F$$

$$\frac{dp^\mu}{d\tau} = \frac{1}{\sqrt{1-\beta^2}}\frac{d}{dt}(mc, mv_x, mv_y, mv_z) \quad \text{—⑫}$$

$$p^\mu = m_0 \frac{dx^\mu}{d\tau} = \frac{m_0}{\sqrt{1-\beta^2}} \frac{dx^\mu}{dt} \quad \text{—①}$$

$x^\mu = (ct, x, y, z)$ なので

$$p^\mu = \left(\frac{m_0}{\sqrt{1-\beta^2}} \frac{d(ct)}{dt}, \frac{m_0}{\sqrt{1-\beta^2}} \frac{dx}{dt}, \frac{m_0}{\sqrt{1-\beta^2}} \frac{dy}{dt}, \frac{m_0}{\sqrt{1-\beta^2}} \frac{dz}{dt} \right)$$

$$p^\mu = \left(\frac{m_0 c}{\sqrt{1-\beta^2}}, \frac{m_0}{\sqrt{1-\beta^2}} \frac{dx}{dt}, \frac{m_0}{\sqrt{1-\beta^2}} \frac{dy}{dt}, \frac{m_0}{\sqrt{1-\beta^2}} \frac{dz}{dt} \right) \quad \text{—②}$$

空間部分について $(j=1,2,3)$

$\frac{d}{dt}(p^j) = F^j$ とおくと

$$\frac{d}{d\tau} p^j = \frac{1}{\sqrt{1-\beta^2}} \frac{dp^j}{dt} = \frac{F^j}{\sqrt{1-\beta^2}} \quad (j=1,2,3) \quad \text{—③}$$

$$p^\mu = \left(\frac{m_0 c}{\sqrt{1-\beta^2}}, \frac{m_0 v_x}{\sqrt{1-\beta^2}}, \frac{m_0 v_y}{\sqrt{1-\beta^2}}, \frac{m_0 v_z}{\sqrt{1-\beta^2}} \right)$$

$$\frac{dp^j}{d\tau} = \frac{1}{\sqrt{1-\beta^2}} \frac{d}{dt} p^j$$

$$\frac{d}{dt}\left(\frac{m_0}{\sqrt{1-\beta^2}}\right) = m_0 \frac{d}{dt}(1-\beta^2)^{-\frac{1}{2}} = \frac{-m_0}{2}(1-\beta^2)^{-\frac{3}{2}} \times \left(-\frac{2}{c^2} v_x \frac{dv_x}{dt} - \frac{2}{c^2} v_y \frac{dv_y}{dt} - \frac{2}{c^2} v_z \frac{dv_z}{dt}\right)$$

$$= \frac{-m_0}{2}(1-\beta^2)^{-\frac{3}{2}} \times \left(\frac{-2}{c^2}\right) \times \left(v_x \frac{dv_x}{dt} + v_y \frac{dv_y}{dt} + v_z \frac{dv_z}{dt}\right)$$

$$\frac{d}{dt}\left(\frac{m_0}{\sqrt{1-\beta^2}}\right) = + \frac{m_0}{c^2}(1-\beta^2)^{-\frac{3}{2}} \cdot \vec{v} \cdot \frac{d\vec{v}}{dt}$$

$$c^2 \frac{d}{dt}\left(\frac{m_0}{\sqrt{1-\beta^2}}\right) = m_0 (1-\beta^2)^{-\frac{3}{2}} \vec{v} \cdot \frac{d\vec{v}}{dt} = \frac{m_0}{(\sqrt{1-\beta^2})^3} \vec{v} \cdot \frac{d\vec{v}}{dt}$$

(次ページへ)

$$\rightarrow \frac{d}{dt}\left(\frac{m_0 c^2}{\sqrt{1-\beta^2}}\right) = \frac{m_0}{(\sqrt{1-\beta^2})^3} \vec{v} \cdot \frac{d\vec{v}}{dt} = \vec{F} \cdot \vec{v}$$

$$\frac{d}{dt}\left(\frac{m_0}{\sqrt{1-\beta^2}}\vec{v}\right)=\vec{F}$$

$$\vec{F}\cdot\vec{v} = \vec{v}\cdot\frac{d}{dt}\left(\frac{m_0}{\sqrt{1-\beta^2}}\vec{v}\right)$$

$$=\vec{v}\,\frac{d}{dt}\left(\frac{m_0}{\sqrt{1-\beta^2}}\right)\cdot\vec{v}+\vec{v}\,\frac{m_0}{\sqrt{1-\beta^2}}\frac{d\vec{v}}{dt}$$

$$=\vec{v}\left\{\frac{m_0}{c^2}(1-\beta^2)^{-\frac{3}{2}}\vec{v}\cdot\frac{d\vec{v}}{dt}\right\}\vec{v}+\frac{m}{\sqrt{1-\beta^2}}\vec{v}\cdot\frac{d\vec{v}}{dt}$$

$$=\frac{m_0}{c^2}(1-\beta^2)^{-\frac{3}{2}}v^2\frac{d\vec{v}}{dt}\vec{v}+\frac{m_0}{\sqrt{1-\beta^2}}\vec{v}\frac{d\vec{v}}{dt}$$

$$=\frac{m_0}{\sqrt{1-\beta^2}}\cdot\frac{1}{(1-\beta^2)}\frac{v^2}{c^2}\frac{d\vec{v}}{dt}\vec{v}+\frac{m_0}{\sqrt{1-\beta^2}}\vec{v}\cdot\frac{d\vec{v}}{dt}$$

$$\frac{m_0}{\sqrt{1-\beta^2}}\frac{1}{(1-\beta^2)}\beta^2\cdot\vec{v}\frac{d\vec{v}}{dt}+\frac{m_0}{\sqrt{1-\beta^2}}\vec{v}\cdot\frac{d\vec{v}}{dt}$$

$$=\frac{m_0}{\sqrt{1-\beta^2}}\left\{\frac{1}{1-\beta^2}\beta^2+1\right\}\vec{v}\frac{d\vec{v}}{dt}$$

$$=\frac{m_0}{\sqrt{1-\beta^2}}\left\{\frac{\beta^2}{1-\beta^2}+\frac{1-\beta^2}{1-\beta^2}\right\}\vec{v}\frac{d\vec{v}}{dt}$$

$$=\frac{m_0}{\sqrt{1-\beta^2}}\frac{1}{1-\beta^2}\vec{v}\frac{d\vec{v}}{dt}=\frac{m_0}{(\sqrt{1-\beta^2})^3}\vec{v}\cdot\frac{d\vec{v}}{dt}$$

(前ページより)

$$\vec{F}\cdot\vec{v}=\frac{m_0}{(\sqrt{1-\beta^2})^3}\vec{v}\frac{d\vec{v}}{dt}=\frac{d}{dt}\left(\frac{m_0c^2}{\sqrt{1-\beta^2}}\right)$$

$$\therefore\ \frac{\vec{F}}{\sqrt{1-\beta^2}}\cdot\vec{v}=\frac{1}{\sqrt{1-\beta^2}}\frac{d}{dt}\left(\frac{m_0c^2}{\sqrt{1-\beta^2}}\right)$$

$$=\frac{d}{d\tau}\left(\frac{m_0c^2}{\sqrt{1-\beta^2}}\right)=\cancel{c\frac{d}{d\tau}\left(\frac{E}{c}\right)}$$

$$\therefore\ \frac{\vec{F}\cdot\vec{v}}{c\sqrt{1-\beta^2}}=\frac{d}{d\tau}\left(\frac{m_0c}{\sqrt{1-\beta^2}}\right)$$

$E=\dfrac{m_0c^2}{\sqrt{1-\beta^2}}$ とおくと $\longrightarrow$ $\vec{F}\cdot\vec{v}=\dfrac{dE}{dt}$ 　 $\dfrac{E}{c}=\dfrac{m_0c}{\sqrt{1-\beta^2}}$ なので

（Eは仕事の次元 すなわち エネルギーを表すものと理解出来る）

$$p^\mu=\left(\frac{E}{c},\ \frac{m_0}{\sqrt{1-\beta^2}}\frac{dx}{dt},\ \frac{m_0}{\sqrt{1-\beta^2}}\frac{dy}{dt},\ \frac{m_0}{\sqrt{1-\beta^2}}\frac{dz}{dt}\right)$$

$$p^\mu=\left(\frac{E}{c},\ p_x,\ p_y,\ p_z\right)$$

$$\begin{cases}p_x=m\dfrac{dx}{dt}\\ p_y=m\dfrac{dy}{dt}\\ p_z=m\dfrac{dz}{dt}\end{cases}\qquad m=\frac{m_0}{\sqrt{1-\beta^2}}$$

p^μは四元運動量（ローレンツ変換に従う物理量）

$p^\mu p_\mu=$一定（スカラー）である（ローレンツ変換に従う物理量はローレンツ積は一定である）

$$\frac{E^2}{c^2}-p_x^2-p_y^2-p_z^2=(\text{一定})$$

$$\frac{E^2}{c^2}-p^2=\frac{E_0^2}{c^2}\qquad E_0=m_0c^2$$

$$\frac{E^2}{c^2}-p^2=\frac{m_0^2c^4}{c^2}=m_0^2c^2$$

$$\frac{E^2}{c^2}=p^2+m_0^2c^2$$

$$E^2=c^2p^2+m_0^2c^4$$

$$E=\sqrt{c^2p^2+m_0^2c^4}$$

$m_0=0$ のときは　$E=cp$　$p=\dfrac{E}{c}$