$$x^\mu = \left(x^0 \quad x^1 \quad x^2 \quad x^3\right) \quad x^\mu = \left(ct \quad x \quad y \quad z\right) \qquad x^\mu = \left(ct \quad \mathbf{x}\right)$$

$$x_\mu = \left(x^0 \quad -x^1 \quad -x^2 \quad -x^3\right)$$

$$x_\mu = \left(x_0 \quad x_1 \quad x_2 \quad x_3\right) \quad x_\mu = \left(ct \quad -x \quad -y \quad -z\right) \quad x_\mu = \left(ct \quad -\mathbf{x}\right)$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\partial_{ct} \quad \nabla\right)$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\partial_{ct} \quad -\nabla\right)$$

$$\Lambda^\mu_\nu = \Lambda^\nu_\mu = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \beta = \frac{v}{c} \quad d\tau = \sqrt{1-\beta^2}\,dt$$

Lorentz boost along x-axis $\quad x'^\mu = \Lambda^\mu_\nu x^\nu \quad x'_\mu = \Lambda^\nu_\mu x_\nu$

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad g^{\mu\nu} = g_{\mu\nu} = diag\left(1 \quad -1 \quad -1 \quad -1\right)$$ 対角行列

$$a^\mu = g^{\mu\nu} a_\nu \quad a_\mu = g_{\mu\nu} a^\nu$$

ローレンツ不変量

内積の定義 → 4元ベクトル **a** と **b** の内積を次のように定義する。

Lorentz scalar $\quad a \cdot b = a^\mu b_\mu = a^0 b^0 - \mathbf{a} \cdot \mathbf{b} = a_\mu b^\mu$

$a^2 > 0$ time-like 4-vector

$a^2 = 0$ light-like 4-vector

$a^2 < 0$ space-like 4-vector

$$ds^2 = \left[cdt \quad dx \quad dy \quad dz\right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} cdt \\ dx \\ dy \\ dz \end{bmatrix} = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt^2\left(1 - \frac{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]}{c^2}\right) = c^2dt^2\left(1 - \frac{v^2}{c^2}\right)$$

$$d\tau^2 = \frac{ds^2}{c^2} = dt^2\left(1 - \frac{v^2}{c^2}\right) = \left(1 - \beta^2\right)dt^2 \qquad d\tau = \frac{ds}{c} = \sqrt{1-\beta^2}\,dt \rightarrow$$ 固有時（ローレンツスカラー量）

4-momentum $\quad p^\mu = \begin{pmatrix} \frac{E}{c} & \mathbf{p} \end{pmatrix} \quad p_\mu = \begin{pmatrix} \frac{E}{c} & -\mathbf{p} \end{pmatrix}$

Invariant mass $\quad p^\mu p_\mu = p_\mu p^\mu = \frac{E^2}{c^2} - \mathbf{p}^2 = m^2c^2$

Particle velocity $\quad \gamma = \frac{E}{mc^2} \quad \beta = \frac{|\mathbf{p}|c}{E}$

$$\frac{E^2}{c^2} - \mathbf{p}^2 = m^2c^2 \quad \frac{E^2}{c^2} = m^2c^2 + \mathbf{p}^2 \quad E^2 = m^2c^4 + \mathbf{p}^2c^2 \qquad E = \sqrt{m^2c^4 + \mathbf{p}^2c^2}$$

★４元ベクトル $\mathbf{a}$ と $\mathbf{b}$ のローレンツ変換後の内積をとると

$$\mathbf{a'} \cdot \mathbf{b'} = a'^\mu b'_\mu = a'^0 b'^0 - a'^1 b'^1 - a'^2 b'^2 - a'^3 b'^3$$

$$a' \cdot b' = a'^\mu b'_\mu = (\gamma a^0 - \beta\gamma a^1)(\gamma b^0 - \beta\gamma b^1) - (-\beta\gamma a^0 + \gamma a^1)(-\beta\gamma b^0 + \gamma b^1) - a^2b^2 - a^3b^3$$

$$= (\gamma a^0 - \beta\gamma a^1)(\gamma b^0 - \beta\gamma b^1) - (-\beta\gamma a^0 + \gamma a^1)(-\beta\gamma b^0 + \gamma b^1) - a^2b^2 - a^3b^3$$
$$= \gamma^2 a^0b^0 - \gamma^2\beta^2 a^0b^0 + \beta^2\gamma^2 a^1b_1^1 - \gamma^2 a^1b^1 - a^2b^2 - a^3b^3$$
$$= \gamma^2(1-\beta^2)a^0b^0 - \gamma^2(1-\beta^2)a^1b^1 - a^2b^2 - a^3b^3$$
$$= \frac{1}{(1-\beta^2)}(1-\beta^2)a^0b^0 - \frac{1}{(1-\beta^2)}(1-\beta^2)a^1b^1 - a^2b^2 - a^3b^3$$
$$= a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3 = a^\mu b_\mu = \mathbf{a} \cdot \mathbf{b}$$

∴　ローレンツ変換について内積は不変量になる。

$a' = La$
$b' = Lb$

$$a' \cdot b' = (a')^t G(b') = (La)^t G(Lb) = a^t L^t GLb = a^t (L^t GL)b = a^t Gb = a \cdot b$$

$$L^t GL = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ \beta\gamma & -\gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ \beta\gamma & -\gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \gamma^2 - \beta^2\gamma^2 & -\beta\gamma^2 + \beta\gamma^2 & 0 & 0 \\ -\beta\gamma^2 + \beta\gamma^2 & \beta^2\gamma^2 - \gamma^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma^2 - \beta^2\gamma^2 & -\beta\gamma^2 + \beta\gamma^2 & 0 & 0 \\ -\beta\gamma^2 + \beta\gamma^2 & \beta^2\gamma^2 - \gamma^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \gamma^2(1-\beta^2) & 0 & 0 & 0 \\ 0 & -\gamma^2(1-\beta^2) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$\therefore$

$$L^t GL = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = G$$