[遅延ポテンシャル]

$$\begin{cases} \varepsilon_0 \mu_0 \dfrac{\partial^2 \phi}{\partial t^2} - \Delta \phi = \dfrac{\rho}{\varepsilon_0} & \text{—①} \\ \varepsilon_0 \mu_0 \dfrac{\partial^2 \vec{A}}{\partial t^2} - \Delta \vec{A} = \mu_0 \vec{j} & \text{—②} \end{cases}$$

フーリエ変換より

$$\begin{cases} \rho(x,y,z,t) = \displaystyle\int \rho_\omega (x,y,z,\omega) e^{-i\omega t} d\omega & \text{—③} \\ \phi(x,y,z,t) = \displaystyle\int \phi_\omega (x,y,z,\omega) e^{-i\omega t} d\omega & \text{—④} \end{cases}$$

$$\frac{\partial^2 \phi(x,y,z,t)}{\partial t^2} = \int \phi_\omega(x,y,z,\omega) \frac{\partial^2}{\partial t^2}\left(e^{-i\omega t}\right) d\omega$$

$$= -\omega^2 \int \phi_\omega(x,y,z,\omega) e^{-i\omega t} d\omega \quad \text{—⑤}$$

$$\Delta \phi(x,y,z,t) = \int \Delta \phi_\omega(x,y,z,\omega) e^{-i\omega t} d\omega \quad \text{—⑥}$$

①に代入する

$$\varepsilon_0 \mu_0 \frac{\partial^2 \phi}{\partial t^2} = \int \left\{ -\varepsilon_0 \mu_0 \omega^2 \phi_\omega (x,y,z,\omega) \right\} e^{-i\omega t} d\omega \quad \text{—⑦}$$

$$-\Delta \phi = \int (-\Delta \phi_\omega) e^{-i\omega t} d\omega \quad \text{—⑧}$$

$$\left(\varepsilon_0 \mu \frac{\partial^2 \phi}{\partial t^2} - \Delta \phi \right) = \int \left\{ -\varepsilon_0 \mu_0 \omega^2 \phi_\omega - \Delta \phi_\omega \right\} e^{-i\omega t} d\omega \quad \text{—⑨}$$

$$\frac{\rho}{\varepsilon_0} = \int \frac{\rho_\omega(x,y,z,\omega)}{\varepsilon_0} e^{-i\omega t} d\omega \quad —⑩$$

⑨ = ⑩ より

$$\therefore \frac{\rho_\omega}{\varepsilon_0}(x,y,z,\omega) = (-\varepsilon_0\mu_0\omega^2 - \Delta)\phi_\omega$$

$$\therefore -(\varepsilon_0\mu_0\omega^2 + \Delta)\phi_\omega = \frac{\rho_\omega}{\varepsilon_0}(x,y,z,\omega)$$

$$(\varepsilon_0\mu_0\omega^2 + \nabla^2)\phi_\omega = -\frac{\rho_\omega}{\varepsilon_0} \quad —⑪$$

mといまはおいて計算する

$\varepsilon_0\mu_0\omega^2 = k^2$とおく

$$\therefore k = m = \sqrt{\varepsilon_0\mu_0}\,\omega$$

$$m = k = \frac{\omega}{c}$$

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$$

$$(m^2 + \nabla^2)\phi_\omega = -\frac{\rho_\omega}{\varepsilon_0} \quad —⑫$$

$(m^2 + \nabla^2)\, G(\vec{r}) = \delta(\vec{r})$ とおけば —⑬

$(m^2 + \nabla^2) G_m(x,y,z) = \delta(x)\delta(y)\delta(z)$ を満たすGが求まると

$$\phi_\omega(x,y,z) = \int G_m(x-\xi, y-\eta, z-\zeta)\left(-\frac{\rho_\omega}{\varepsilon_0}(\xi,\eta,\zeta)\right) d\xi d\eta d\zeta$$

$$= -\frac{1}{\varepsilon_0}\int G(x-\xi, y-\eta, z-\zeta)\rho_\omega(\xi,\eta,\zeta)\, d\xi d\eta d\zeta$$

が求める解である —⑭

フーリエ変換より

$$G(x,y,z) = \frac{1}{(2\pi)^3}\int \tilde{G}(k_x, k_y, k_z)\, e^{i\vec{k}\cdot\vec{r}}\, dk_x dk_y dk_z$$

$$e^{i\vec{k}\cdot\vec{r}} = e^{i(k_x x + k_y y + k_z z)} \quad —⑮$$

⑮と⑬に代入する

$$\cancel{(m^2 + \nabla^2) G_k(x,y,z) = \frac{1}{(2\pi)^3}\int \tilde{G}_k(k_x,k_y,k_z)\, e^{i\vec{\xi}\cdot\vec{r}}\, d\vec{\xi}\; d\xi_x d\xi_y d\xi_z}$$

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$$(m^2+\nabla^2)\,G_m(\vec{r})=\delta(\vec{r})$$

フーリエ変換より

$$G_m(\vec{r})=\frac{1}{(2\pi)^3}\int \tilde{G}_m(k_x,k_y,k_z)\,e^{i\vec{k}\vec{r}}\,dk_x dk_y dk_z$$

$$e^{i\vec{k}\vec{r}}=e^{i(k_x x+k_y y+k_z\cdot z)} \quad \text{—(15)}$$

$$(m^2+\nabla^2)\,G_m(\vec{r})=\frac{1}{(2\pi)^3}\int \tilde{G}_m(\vec{k})\{m^2-k^2\}e^{i\vec{k}\vec{r}}\,dk_x dk_y dk_z$$

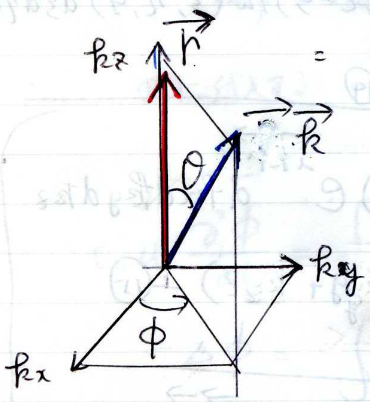
$$=\underbrace{\frac{1}{(2\pi)^3}\int e^{i\vec{k}\cdot\vec{r}}\,dk_x dk_y dk_z}_{\delta(x)\delta(y)\delta(z)} \quad \text{—(16)}$$

$$\therefore\ \tilde{G}_m(\vec{k})(m^2-k^2)=1$$

$$\tilde{G}_m(k)=\frac{1}{m^2-k^2} \quad \text{—(17)}$$

$$\therefore\ G_m(\vec{r})=\frac{1}{(2\pi)^3}\int \tilde{G}(\vec{k})\,e^{i\vec{k}\cdot\vec{r}}\,dk_x dk_y dk_z$$

$$=\frac{1}{(2\pi)^3}\iiint \frac{e^{i\vec{k}\cdot\vec{r}}}{m^2-k^2}\,dk_x dk_y dk_z \quad \text{—(18)}$$



$$k_x=k\sin\theta\cos\phi$$
$$k_y=k\sin\theta\sin\phi$$
$$k_z=k\cos\theta$$

$$dk_x dk_y dk_z=k^2\sin\theta\,d\phi\,d\theta$$

$$G_m(\vec{r})=\frac{1}{(2\pi)^3}\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{e^{i\vec{k}\cdot\vec{r}}}{m^2-k^2}\,k^2\sin\theta\,d\phi\,d\theta \quad \text{—(19)}$$

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$$(m^2+\nabla^2)\,G_m(\vec{r}) = \delta(\vec{r})$$

フーリエ変換より

$$G_m(\vec{r}) = \frac{1}{(2\pi)^3}\int \widetilde{G}_m(k_x,k_y,k_z)\,e^{i\vec{k}\vec{r}}\,dk_x dk_y dk_z$$

$$e^{i\vec{k}\vec{r}} = e^{i(k_x x + k_y y + k_z z)} \quad \text{—⑮}$$

$$(m^2+\nabla^2)\,G_m(\vec{r}) = \frac{1}{(2\pi)^3}\int \widetilde{G}_m(\vec{k})\{m^2-k^2\}\,e^{i\vec{k}\vec{r}}\,dk_x dk_y dk_z$$

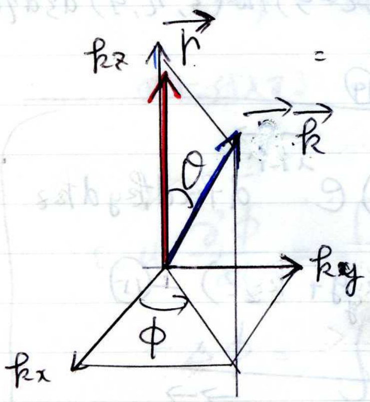
$$= \underbrace{\frac{1}{(2\pi)^3}\int e^{i\vec{k}\cdot\vec{r}}\,dk_x dk_y dk_z}_{\delta(x)\delta(y)\delta(z)} \quad \text{—⑯}$$

$$\therefore\ \widetilde{G}_m(\vec{k})(m^2-k^2) = 1$$

$$\widetilde{G}_m(k) = \frac{1}{m^2-k^2} \quad \text{—⑰}$$

$$\therefore\ G_m(\vec{r}) = \frac{1}{(2\pi)^3}\int \widetilde{G}(\vec{k})\,e^{i\vec{k}\cdot\vec{r}}\,dk_x dk_y dk_z$$

$$= \frac{1}{(2\pi)^3}\iiint \frac{e^{i\vec{k}\cdot\vec{r}}}{m^2-k^2}\,dk_x dk_y dk_z \quad \text{—⑱}$$



$$k_x = k\sin\theta\cos\phi$$
$$k_y = k\sin\theta\sin\phi$$
$$k_z = k\cos\theta$$

$$dk_x dk_y dk_z = k^2\sin\theta\,d\phi\,d\theta$$

$$G_m(\vec{r}) = \frac{1}{(2\pi)^3}\int_0^\infty\int_0^\pi\int_0^{2\pi} \frac{e^{i\vec{k}\cdot\vec{r}}}{m^2-k^2}\,k^2\sin\theta\,d\phi\,d\theta \quad \text{—⑲}$$

$$G_m(\vec{r}) = \frac{1}{(2\pi)^2}\int_0^\infty\int_0^\pi \frac{e^{ikr\cos\theta}}{m^2-k^2} k^2 \sin\theta \, d\theta \, dk \quad —⑳$$

$\cos\theta$ $\qquad d(\cos\theta) = \qquad = -\sin\theta \, d\theta$

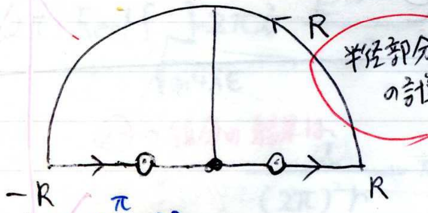
$$= \frac{-1}{(2\pi)^2}\int_0^\infty\int_1^{-1} \frac{e^{ikr\cos\theta}}{m^2-k^2} k^2 \, d(\cos\theta) \, dk \quad —㉑$$

$$= \frac{1}{(2\pi)^2}\int_0^\infty\int_{-1}^{1} \frac{(e^{ikr\cos\theta})}{m^2-k^2} k^2 \, d(\cos\theta) \, dk \quad —㉓$$

$$= \frac{1}{(2\pi)^2}\int_0^\infty \frac{(e^{ikr}-e^{-ikr})}{(m^2-k^2)ikr} k^2 \, dk$$

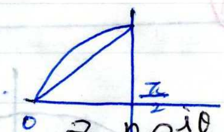
$$= \frac{1}{(2\pi)^2 i}\int_0^\infty \frac{(e^{ikr}-e^{-ikr})}{m^2-k^2} k \, dk$$

$$= \frac{1}{(2\pi)^2 ir}\int_{-\infty}^\infty \frac{e^{ikr}}{(m^2-k^2)} k \, dk \quad —㉓$$



半径部分の計算

$$\int_{-\infty}^{\infty} \frac{k e^{ikr}}{m^2-k^2} k \, dk$$



$z = Re^{i\theta}$

$dz = Ri e^{i\theta} d\theta$

$= iz \, d\theta$

$$\int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi}x} dx$$

$$\int \frac{z e^{izr} z}{m^2-z^2} dz$$

$$-\frac{\pi}{2R}\left[e^{-1}-1\right] \qquad i\int \frac{z^2 e^{iz} z}{m^2-z^2} d\theta$$

$iR(\cos\theta + i\sin\theta)$

$e^{-R\frac{2}{\pi}x}$ $\quad e^{-\frac{2R}{\pi}x}$

$$= \frac{\pi}{2R}\left(1-\frac{1}{e}\right) \frown \left| i\frac{z^2 e^{iz}}{m^2-z^2}\pi \right| \leq \pi \frac{R^2 e^{iR\cos\theta} e^{-R\sin\theta}}{m^2-R^2} \longrightarrow 0$$

$$\frac{\pi R^2}{m^2-R^2} \times \frac{\pi}{2R}\left(1-\frac{1}{e}\right) \to 0$$

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$$\int \frac{z^2 e^{izr}}{m^2-z^2}dz$$

$$\frac{z^2}{m^2-z^2} = \frac{1}{2}$$

$$\frac{1}{2}\int \frac{z e^{izr}}{m-z}dz - \frac{1}{2}\int \frac{z e^{izr}}{m+z}dz$$

$$\cancel{z-m = Re^{i\theta}}$$

$$z-m = Re^{i\theta}$$

$$dz = iRe^{i\theta}d\theta = iz\,d\theta$$

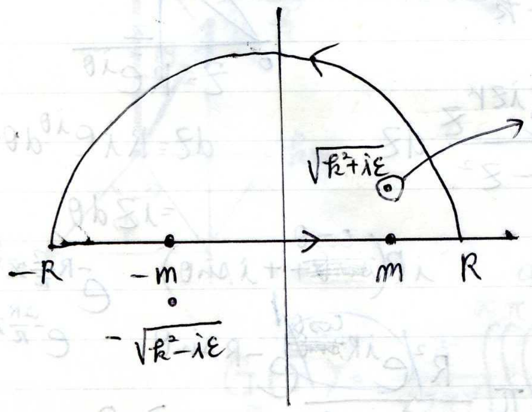
$$\int \frac{(m+Re^{i\theta})}{Re^{i\theta}} e^{iR(m+Re^{i\theta})}$$

$$= e^{im} e^{iR\cos\theta} - e^{-R\sin\theta}$$

$$e^{R}$$

$$Re(f, m) = (z-m)\frac{z e^{izr}}{(m-z)}$$

$$= -z e^{izr} = -m e^{imr}$$



ただひとつの極

近似を用いる

$$G_m(\vec{r}) = \frac{-i}{(2\pi)^2 r}\int_{-\infty}^{\infty}\frac{e^{ikr}k}{m^2-k^2}dk$$

$$= \frac{i}{(2\pi)^2 r}\int_{-\infty}^{\infty}\frac{e^{ikr}k}{k^2-m^2}dk \quad \text{—㉔}$$

ここで

$$\frac{i}{(2\pi)^2 r}\int_{-\infty}^{\infty}\frac{e^{ikr}k}{k^2-m^2-i\varepsilon}dk \quad \text{—㉕}$$

ε はとても小さい。 $\varepsilon>0$ と決めておく

$$k^2-(m^2+i\varepsilon)=0 \qquad k=\pm\sqrt{m^2+i\varepsilon}$$

留数定理より

$$\lim_{k\to\sqrt{m^2+i\varepsilon}}(k-\sqrt{m^2+i\varepsilon})\frac{e^{ikr}k}{(k+\sqrt{m^2+i\varepsilon})(k-\sqrt{m^2+i\varepsilon})}$$

$$=\frac{\sqrt{m^2+i\varepsilon}\,e^{i\sqrt{m^2+i\varepsilon}\,r}}{2\sqrt{m^2+i\varepsilon}}$$

ここで $\varepsilon\to 0$ より $\dfrac{me^{imr}}{2m}=\dfrac{e^{imr}}{2}$ —㉖

$2\pi i\,\mathrm{Res}[f,\sqrt{m^2+i\varepsilon}]=2\pi i\dfrac{e^{imr}}{2}=\pi i e^{imr}$ を代入

㉔の積分の結果は

$$\frac{i}{(2\pi)^2 r}\pi i e^{imr} = -\frac{e^{imr}}{4\pi r} \quad \text{—㉗}$$

ここで $m=k$ とおきなおすと $\qquad k=\dfrac{\omega}{c}$

$$\boxed{G(\vec{r})=-\frac{e^{ikr}}{4\pi r}} \quad \text{—㉘}$$

㉘を⑲に代入すると　　グリーン関数

$$\phi_\omega(\vec{r}) = -\frac{1}{\varepsilon_0}\int\left(-\frac{e^{ikr}}{4\pi r}\right)\rho_\omega(\xi,\eta,\zeta)\,d\xi d\eta d\zeta$$

$$= \frac{1}{4\pi\varepsilon_0}\int\frac{e^{ikr}}{r}\rho_\omega(\xi,\eta,\zeta)\,d\xi d\eta d\zeta \quad \text{—(29)}$$

ここで

$$r = \sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}$$

∴ ⑲に代入すると

$$\phi(x,y,z,t) = \frac{1}{4\pi\varepsilon_0}\iint\frac{e^{ikr}}{r}\rho_\omega(\xi,\eta,\zeta,\omega)e^{-i\omega t}\,d\xi d\eta d\zeta$$

$$= \frac{1}{4\pi\varepsilon_0}\iint\frac{\rho_\omega(\xi,\eta,\zeta,\omega)}{r}e^{i(kr-\omega t)}\,d\xi d\eta d\zeta\, d\omega$$

$$= \frac{1}{4\pi\varepsilon_0}\int\frac{\rho_\omega(\xi,\eta,\zeta,\omega)}{r}e^{-i\omega(t-\frac{k}{\omega}r)}\,d\omega\, d\xi d\eta d\zeta$$

$$\frac{k}{\omega} = \frac{1}{c}$$

$$\phi(x,y,z,t) = \frac{1}{4\pi\varepsilon_0}\iint\frac{\rho_\omega(\xi,\eta,\zeta,\omega)e^{-i\omega(t-\frac{r}{c})}}{r}\,d\omega\, d\xi d\eta d\zeta$$

③式と比較して

$$\phi(x,y,z,t) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\xi,\eta,\zeta,t-\frac{r}{c})}{r}\,d\xi d\eta d\zeta$$

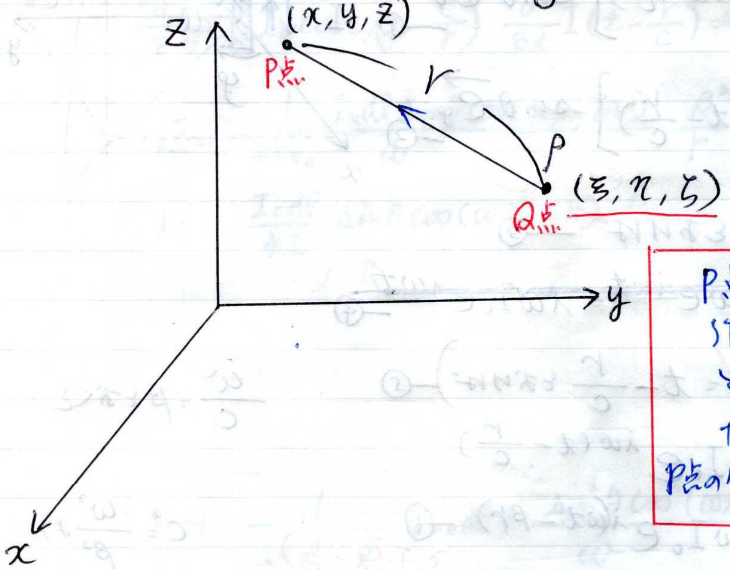
$$\boxed{\rho(\xi,\eta,\zeta,t-\frac{r}{c}) = \int\rho_\omega(\xi,\eta,\zeta,\omega)e^{-i\omega(t-\frac{r}{c})}\,d\omega}$$

$\Delta t = \frac{r}{c}$ だけ以前の電荷が問題になる

電荷の影響が光速 c で空間を伝わって来ることを示す。

同様に

$$\vec{A}(x,y,z,t)=\frac{\mu_0}{4\pi}\int\frac{\vec{i}\left(\xi,\eta,\zeta,t-\frac{r}{c}\right)}{r}d\xi\, d\eta\, d\zeta$$



P点の電位がQ点の電荷により作られ また P点の時間を t [s] とすると 空間を伝わる時間 $\frac{r}{c}$ だけ過去のQ点の電荷がP点の電位を形成している

⇓

$\boxed{\text{遅延ポテンシャル}}$