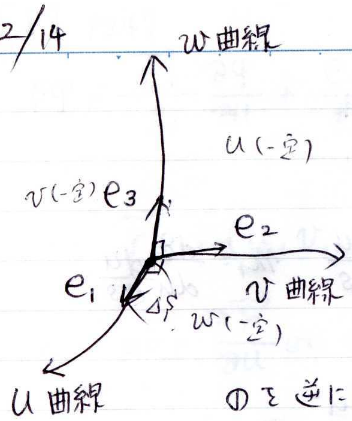
2/14



$u = u(x, y, z)$ $v = v(x, y, z)$ $w = w(x, y, z)$ —①

$(x, y, z) \overset{1対1}{\rightleftarrows} (u, v, w)$ —③

①を逆に解いて

$x = x(u, v, w)$ $y = y(u, v, w)$ $z = z(u, v, w)$

u方向のベクトル ⟶ (uのみが増加する) A B 線上にある

$$\vec{u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

$$e_1 = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}}\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = \frac{\vec{u}}{h_1}$$

$$\boxed{h_1 = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}}$$

$$\nabla u = \frac{\partial u}{\partial x} i + \frac{\partial u}{\partial y} j + \frac{\partial u}{\partial z} k$$ (u平面に対して垂直, uの増加する向き)

u平面の法線ベクトル $\vec{n}_1$
(u=一定)

$$\vec{n}_1 = \frac{\nabla u}{|\nabla u|} = \frac{\nabla u}{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2}}$$

$$|\nabla u| = \frac{\Delta u}{\Delta s}$$ $$\nabla u = |\mathrm{grad}\, u| \cdot \vec{n}_1$$

法線方向の微係数

$$\nabla u = |\mathrm{grad}\, u| \cdot \frac{\vec{u}}{h_1}$$

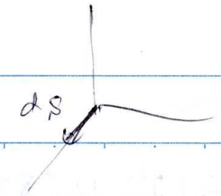
~~$e_1 \times \nabla u = 0$ ならば 垂直~~

$$\cancel{\left(\frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial z} - \frac{\partial z}{\partial u} \frac{\partial u}{\partial y}\right)}$$

$$h_1 = |\mathrm{grad}\, u|$$

No. 26

Date



$\frac{du}{ds}$

$l_1 = \frac{dx}{ds} = \frac{dx}{du} \cdot \frac{du}{ds}$ $\quad m_1 = \frac{d}{du} \cdot \frac{du}{ds}$ $\quad n_1 = \frac{d}{du} \cdot \frac{du}{ds}$

$$ds = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 du^2 + \left(\frac{\partial y}{\partial u}\right)^2 du^2 + \left(\frac{\partial z}{\partial u}\right)^2 du^2}$$

$$= \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}\, du$$

ds

$\therefore \frac{ds}{du} = h_1$ $\quad \therefore \frac{du}{ds} = \frac{1}{h_1}$

$l_1 = \frac{1}{h_1}\frac{\partial x}{\partial u}$ $\quad m_1 = \frac{1}{h_1}\frac{\partial y}{\partial u}$ $\quad n_1 = \frac{1}{h_1}\frac{\partial z}{\partial u}$

$$e_1 = \frac{1}{h_1}\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

u方向の s

$\frac{\frac{\partial x}{\partial u}}{\frac{\partial u}{\partial x}}$ $\quad \overline{\partial x}$ $\quad \frac{\partial x}{\partial u} \times \frac{\partial}{\partial u}$

$\frac{ds}{du} = h_1$ $\quad \frac{du}{ds} = \frac{1}{h_1}$

~~∇uは [illegible] uの増加率 |∇φ| |∇u|~~

法線方向の

$\cancel{|\nabla u| = \frac{1}{h_1}}$

$\nabla u = \frac{1}{h_1}\vec{u}$ $\quad \vec{u} = h_1 \cdot \nabla u$

(u, v, w)

[8] P247

$$\nabla\varphi = \frac{\vec{e_1}}{h_1}\frac{\partial\varphi}{\partial u} + \frac{\vec{e_2}}{h_2}\frac{\partial\varphi}{\partial v} + \frac{\vec{e_3}}{h_3}\frac{\partial\varphi}{\partial w}$$ (証明せよ)

$\vec{r} = \vec{r}(u, v, w)$ より

$$d\vec{r} = \frac{\partial\vec{r}}{\partial u}du + \frac{\partial\vec{r}}{\partial v}dv + \frac{\partial\vec{r}}{\partial w}dw \quad ①$$

$\nabla\varphi = f_1\vec{e_1} + f_2\vec{e_2} + f_3\vec{e_3}$ とおくと ―②

$$d\varphi = \nabla\varphi\cdot d\vec{r} \quad ③$$

$$\frac{\partial\vec{r}}{\partial u} = h_1\vec{e_1} \quad \frac{\partial\vec{r}}{\partial v} = h_2\vec{e_2} \quad \frac{\partial\vec{r}}{\partial w} = h_3\vec{e_3} \quad ④$$

④を①へ代入

$$d\vec{r} = h_1\vec{e_1}du + h_2\vec{e_2}dv + h_3\vec{e_3}dw \quad ⑤$$

$$d\varphi = (f_1\vec{e_1} + f_2\vec{e_2} + f_3\vec{e_3})(h_1du\vec{e_1} + h_2dv\vec{e_2} + h_3dw\vec{e_3}) \quad ⑥$$

$$= f_1h_1du + f_2h_2dv + f_3h_3dw \quad ⑦$$

全微分より

$$= \frac{\partial\varphi}{\partial u}du + \frac{\partial\varphi}{\partial v}dv + \frac{\partial\varphi}{\partial w}dw \quad ⑧$$

⑦⑧の比較

$$\frac{\partial\varphi}{\partial u} = f_1h_1 \quad \frac{\partial\varphi}{\partial v} = f_2h_2 \quad \frac{\partial\varphi}{\partial w} = f_3h_3 \quad ⑨$$

$$f_1 = \frac{1}{h_1}\frac{\partial\varphi}{\partial u} \quad f_2 = \frac{1}{h_2}\frac{\partial\varphi}{\partial v} \quad f_3 = \frac{1}{h_3}\frac{\partial\varphi}{\partial w} \quad ⑩$$

②へ代入すると

$$\nabla\varphi = \frac{1}{h_1}\frac{\partial\varphi}{\partial u}\vec{e_1} + \frac{1}{h_2}\frac{\partial\varphi}{\partial v}\vec{e_2} + \frac{1}{h_3}\frac{\partial\varphi}{\partial w}\vec{e_3}$$ (おわり)

$u(u, v, w)$

$\varphi = u$

(u, v, w)は独立な変数なので

$$\nabla\varphi = \frac{1}{h_1}\frac{\partial\varphi}{\partial u}\vec{e}_1 + \frac{1}{h_2}\frac{\partial\varphi}{\partial v}\vec{e}_2 + \frac{1}{h_3}\frac{\partial\varphi}{\partial w}\vec{e}_3$$

より

$$\varphi = u \text{ とおくと} = \left(\frac{\vec{e}_1}{h_1}\frac{\partial}{\partial u} + \frac{\vec{e}_2}{h_2}\frac{\partial}{\partial v} + \frac{\vec{e}_3}{h_3}\frac{\partial}{\partial w}\right)\varphi$$

$$\nabla u = \frac{1}{h_1}\frac{\partial u}{\partial u}\vec{e}_1 = \frac{1}{h_1}\vec{e}_1$$

$$\nabla v = \frac{1}{h_2}\frac{\partial v}{\partial v}\vec{e}_2 = \frac{1}{h_2}\vec{e}_2$$

$$\nabla w = \frac{1}{h_3}\frac{\partial w}{\partial w}\vec{e}_3 = \frac{1}{h_3}\vec{e}_3$$

$$\boxed{\begin{aligned}\vec{e}_1 &= h_1\nabla u\\ \vec{e}_2 &= h_2\nabla v\\ \vec{e}_3 &= h_3\nabla w\end{aligned}}$$

$$\boxed{h_1 = \frac{1}{|\nabla u|} \quad h_2 = \frac{1}{|\nabla v|} \quad h_3 = \frac{1}{|\nabla w|}}$$

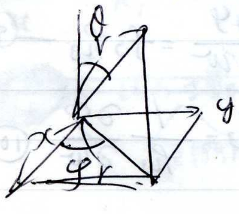
$\vec{e}_1, \vec{e}_2, \vec{e}_3$は単位ベクトルなので

$$|\nabla u| = \frac{du}{dS_1} \longrightarrow dS_1 = \frac{1}{|\nabla u|}du = h_1 du$$

$$|\nabla w| = \frac{dw}{dS_3} \longrightarrow dS_2 = \frac{1}{|\nabla v|}dv = h_2 dv$$

$$|\nabla v| = \frac{dv}{dS_2} \longrightarrow dS_3 = \frac{1}{|\nabla w|}dw = h_3 dw$$

$$dS^2 = dS_1^2 + dS_2^2 + dS_3^2$$



$$x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi$$

$$z = r\cos\theta$$

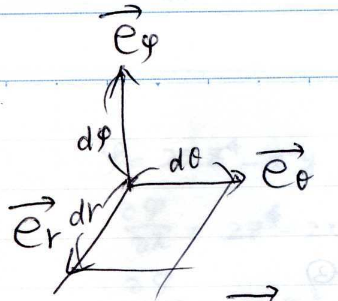
$$\frac{\partial x}{\partial r} = \sin\theta\cos\varphi$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\varphi$$

$$\frac{\partial z}{\partial r} = \cos\theta$$

$$h_1 = \sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta}$$

$$= \left| \frac{1}{h_1 \partial r} \right.$$



$dS_r =$

$$\vec{e}_r = \frac{\nabla u}{h_1}$$

$$\frac{\frac{\partial \vec{r}}{\partial r}}{h_1} = \vec{e}_r$$

$\frac{\partial \vec{r}}{\partial r} = h_1 \vec{e}_r$	$\vec{e}_r = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial r}$
$\frac{\partial \vec{r}}{\partial \theta} = h_2 \vec{e}_\theta$	$\vec{e}_\theta = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \theta}$
$\frac{\partial \vec{r}}{\partial \varphi} = h_3 \vec{e}_\varphi$	$\vec{e}_\varphi = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial \varphi}$

$$\left|\frac{\partial \vec{r}}{\partial r}\right| = h_1 \qquad \left|\frac{\partial \vec{r}}{\partial \theta}\right| = h_2 \qquad \left|\frac{\partial \vec{r}}{\partial \varphi}\right| = h_3$$

$$= 1 \qquad = r \qquad = r\sin\theta$$

$$dS_1 = |\partial \vec{r}| = h_1 dr \qquad dS_2 = |d\vec{r}| = h_2 d\theta \qquad dS_3 = |d\vec{r}| = h_3 d\varphi$$

$$dS_1 = h_1 dr \qquad dS_2 = h_2 d\theta \qquad dS_3 = h_3 d\varphi$$

$$= dr \qquad dS_2 = r\,d\theta \qquad dS_3 = r\sin\theta\, d\varphi$$

$$dV = dS_1 dS_2 \cdot dS_3 = r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$\frac{1}{h_1}\frac{\partial x}{\partial u_1} = h_1 \frac{\partial u_1}{\partial x} \qquad \frac{\partial x}{\partial u_1} = h_1^2 \frac{\partial u_1}{\partial x}$$

$$\frac{\partial y}{\partial u_1} = h_1^2 \frac{\partial u_1}{\partial y}$$

$$\frac{\partial z}{\partial u_1} = h_1^2 \frac{\partial u_1}{\partial z}$$

$$\frac{\partial x}{\partial u_1}\vec{i} + \frac{\partial y}{\partial u_1}\vec{j} + \frac{\partial z}{\partial u_1}\vec{k} = h_1^2\left(\frac{\partial u_1}{\partial x}\vec{i} + \frac{\partial u_1}{\partial y}\vec{j} + \frac{\partial u_1}{\partial z}\vec{k}\right)$$

$$\boxed{\frac{\partial \vec{r}}{\partial u_1} = h_1^2 \nabla u_1 \qquad \frac{\partial \vec{r}}{\partial u_2} = h_2^2 \nabla u_2 \qquad \frac{\partial \vec{r}}{\partial u_3} = h_3^2 \nabla u_3}$$

$\nabla \cdot (A_1 \vec{e_1})$

$$\nabla u_1 = \frac{\vec{e_1}}{h_1} \quad \nabla u_2 = \frac{\vec{e_2}}{h_2} \quad \nabla u_3 = \frac{\vec{e_3}}{h_3}$$

$$\left(\frac{\frac{\partial \vec{r}}{\partial u_1}}{h_1} = \vec{e_1} \quad \frac{\frac{\partial \vec{r}}{\partial u_2}}{h_2} = \vec{e_2} \quad \frac{\frac{\partial \vec{r}}{\partial u_3}}{h_3} = \vec{e_3} \right)$$

$$\frac{\partial \vec{r}}{\partial u_1} = h_1 \vec{e_1} \quad \frac{\partial \vec{r}}{\partial u_2} = h_2 \vec{e_2} \quad \frac{\partial \vec{r}}{\partial u_3} = h_3 \vec{e_3}$$

$$\vec{e_1} = \frac{1}{h_1} \frac{\partial \vec{r}}{\partial u_1} =$$

$$= \frac{1}{h_1} \left(\frac{\partial x}{\partial u_1} \vec{i} + \frac{\partial y}{\partial u_1} \vec{j} + \frac{\partial z}{\partial u_1} \vec{k} \right)$$

$$\vec{e_1} = \left(\frac{1}{h_1} \frac{\partial x}{\partial u_1}, \frac{1}{h_1} \frac{\partial y}{\partial u_1}, \frac{1}{h_1} \frac{\partial z}{\partial u_1} \right)$$

$$\vec{e_2} = \left(\frac{1}{h_2} \frac{\partial x}{\partial u_2}, \frac{1}{h_2} \frac{\partial y}{\partial u_2}, \frac{1}{h_2} \frac{\partial z}{\partial u_2} \right)$$

$$\vec{e_3} = \left(\frac{1}{h_3} \frac{\partial x}{\partial u_3}, \frac{1}{h_3} \frac{\partial y}{\partial u_3}, \frac{1}{h_3} \frac{\partial z}{\partial u_3} \right)$$

$$\vec{e_1} = h_1 \nabla u_1 = \cancel{\left(\frac{h_1 \partial x}{h_1 \partial u_1}, h_1 \partial \right)}$$

$$= \left(h_1 \frac{\partial u_1}{\partial x}, h_1 \frac{\partial u_1}{\partial y}, h_1 \frac{\partial u_1}{\partial z} \right)$$

$$\vec{e_2} = h_2 \nabla u_2 = \left(h_2 \frac{\partial u_2}{\partial x}, h_2 \frac{\partial u_2}{\partial y}, h_2 \frac{\partial u_2}{\partial z} \right)$$

$$\vec{e_3} = h_3 \nabla u_3 = \left(h_3 \frac{\partial u_3}{\partial x}, h_3 \frac{\partial u_3}{\partial x}, h_3 \frac{\partial u_3}{\partial x} \right)$$

$\vec{e_1}\cdot\vec{e_2}=0$

$$\frac{1}{h_1h_2}\left(\frac{\partial x}{\partial u_1}\frac{\partial x}{\partial u_2}+\frac{\partial y}{\partial u_1}\frac{\partial y}{\partial u_2}+\frac{\partial z}{\partial u_1}\frac{\partial z}{\partial u_2}\right)=0$$

$\vec{e_1}\cdot\vec{e_3}=0$

$$\frac{1}{h_1h_3}\left(\frac{\partial x}{\partial u_1}\frac{\partial x}{\partial u_3}+\frac{\partial y}{\partial u_1}\frac{\partial y}{\partial u_3}+\frac{\partial z}{\partial u_1}\frac{\partial z}{\partial u_3}\right)=0$$

$$\vec{e_2}\cdot\vec{e_3}\quad \frac{1}{h_2h_3}\left(\frac{\partial x}{\partial u_2}\frac{\partial x}{\partial u_3}+\frac{\partial y}{\partial u_2}\frac{\partial y}{\partial u_3}+\frac{\partial z}{\partial u_2}\frac{\partial z}{\partial u_3}\right)=0$$

$$\sum_{k=1}^{3}\frac{\partial q_k}{\partial u_i}\frac{\partial q_k}{\partial u_j}=0$$

$q_1=x \quad q_2=y \quad q_3=z$ とおくと

$\vec{e_1}\times\vec{e_2}=\vec{e_3}$

$$\nabla u_1=\frac{\vec{e_1}}{h_1}\quad \nabla u_2=\frac{\vec{e_2}}{h_2}\quad \nabla u_3=\frac{\vec{e_3}}{h_3}$$

$$\nabla u_1\times\nabla u_2=\frac{\vec{e_1}}{h_1}\times\frac{\vec{e_2}}{h_2}=\frac{\vec{e_3}}{h_1\cdot h_2}=\frac{h_3}{h_1\cdot h_2}\nabla u_3$$

$$\nabla u_2\times\nabla u_3=\frac{\vec{e_2}}{h_2}\times\frac{\vec{e_3}}{h_3}=\frac{\vec{e_1}}{h_2\cdot h_3}=\frac{h_1}{h_2\cdot h_3}\nabla u_1$$

$\vec{e_1}=h_2\cdot h_3(\nabla u_2\times\nabla u_3)$

$$\nabla\cdot\vec{A}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\varphi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\varphi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\varphi}{\partial u\ldots}\right.\right.$$

Date

$$\vec{A} = A_1\vec{e_1} + A_2\vec{e_2} + A_3\vec{e_3}$$

$\nabla(A_1\vec{e_1})$

$$= \nabla\left[A_1 h_2 h_3 (\nabla u_2 \times \nabla u_3)\right]$$

$$= \nabla(A_1 h_2 h_3)\cdot(\nabla u_2 \times \nabla u_3) + A_1 h_2 h_3 \nabla(\nabla u_2 \times \nabla u_3)$$

$$= \nabla(A_1 h_2 h_3)\cdot(\nabla u_2 \times \nabla u_3) + A_1 h_2 h_3 \{[\nabla\times(\nabla u_2)]\cdot\nabla u_3 - [\nabla\times(\nabla u_3)]\cdot\nabla u_2\}$$

$$= \nabla(A_1 h_2 h_3)\cdot(\nabla u_2 \times \nabla u_3) = \nabla(A_1 h_2 h_3)\frac{\vec{e_1}}{h_2 h_3}$$

$$= \left(\frac{\vec{e_1}}{h_1}\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\vec{e_2}}{h_2}\frac{\partial}{\partial u_2}(A_1 h_2 h_3) + \frac{\vec{e_3}}{h_3}\frac{\partial}{\partial u_3}(A_1 h_2 h_3)\right)\cdot\frac{\vec{e_1}}{h_2 h_3}$$

$$= \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_1}(A_1 h_2 h_3)$$

$$\nabla(A_2\vec{e_2}) = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_2}(A_2 h_3 h_1)$$

$$\nabla(A_3\vec{e_3}) = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_3}(A_3 h_1 h_2)$$

$$\boxed{\begin{aligned}\nabla\cdot\vec{A} &= \nabla(A_1\vec{e_1} + A_2\vec{e_2} + A_3\vec{e_3})\\ &= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_3 h_1) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\right]\end{aligned}}$$

$$\vec{A} = \nabla\varphi = \frac{\partial\varphi}{h_1\partial u_1}\vec{e_1} + \frac{1}{h_2}\frac{\partial\varphi}{\partial u_2}\vec{e_2} + \frac{1}{h_3}\frac{\partial\varphi}{\partial u_3}\cdot\vec{e_3}$$

とおけば

$$A_1 = \frac{1}{h_1}\frac{\partial\varphi}{\partial u_1} \quad A_2 = \frac{1}{h_2}\frac{\partial\varphi}{\partial u_2} \quad A_3 = \frac{1}{h_3}\frac{\partial\varphi}{\partial u_3}$$

$$\nabla \times (A_1 \vec{e_1})$$

$$\nabla \times (A_1 h_1 \nabla u_1)$$

$$= \nabla(A_1 h_1) \times \nabla u_1 + A_1 h_1 \nabla \times \nabla u_1$$

$$= \left(\vec{e_1} \frac{1}{h_1} \frac{\partial (A_1 h_1)}{\partial u_1} + \vec{e_2} \frac{1}{h_2} \frac{(A_1 h_1)}{\partial u_2} + \vec{e_3} \frac{1}{h_3} \frac{\partial (A_1 h_1)}{\partial u_3} \right\} \times \frac{\vec{e_1}}{h_1}$$

$$= \vec{e_2} \times \vec{e_1} \frac{1}{h_1 h_2} \frac{\partial (A_1 h_1)}{\partial u_2} + \vec{e_3} \times \vec{e_1} \frac{1}{h_3 h_1} \frac{\partial (A_1 h_1)}{\partial u_3}$$

$$= \vec{e_2} \frac{1}{h_3 h_1} \frac{\partial}{\partial u_3} (A_1 h_1) - \vec{e_3} \cdot \frac{1}{h_1 \cdot h_2} \frac{\partial}{\partial u_2} (A_1 h_1) \quad — ①$$

$$\nabla \times (A_2 \vec{e_2})$$

$$= \vec{e_3} \frac{1}{h_1 \cdot h_2} \frac{\partial}{\partial u_1} (A_2 h_2) - \vec{e_1} \frac{1}{h_2 \cdot h_3} \frac{\partial}{\partial u_3} (A_2 h_2) \quad — ②$$

$$\nabla \times (A_2 \vec{e_3})$$

$$= \vec{e_1} \frac{1}{h_2 \cdot h_3} \frac{\partial}{\partial u_2} (A_3 h_3) - \vec{e_2} \frac{1}{h_3 h_1} \frac{\partial}{\partial u_1} (A_3 \cdot h_3) \quad — ③$$

$$\boxed{\begin{aligned} &\nabla \times (A_1 \vec{e_1} + A_2 \vec{e_2} + A_3 \vec{e_3}) \\ &= \vec{e_1} \left\{ \frac{1}{h_2 \cdot h_3} \frac{\partial}{\partial u_2} (A_3 h_3) - \frac{1}{h_2 h_3} \frac{\partial}{\partial u_3} (A_2 h_2) \right\} \\ &\quad + \vec{e_2} \left\{ \frac{1}{h_3 h_1} \frac{\partial}{\partial u_3} (A_1 h_1) - \frac{1}{h_3 h_1} \frac{\partial}{\partial u_1} (A_3 \cdot h_3) \right\} \\ &\quad + \vec{e_3} \left\{ \frac{1}{h_1 h_2} \frac{\partial}{\partial u_1} (A_2 h_2) - \frac{1}{h_1 h_2} \frac{\partial}{\partial u_2} (A_1 h_1) \right\} \end{aligned}}$$

$$\nabla\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}(A_1h_2h_3)+\frac{\partial}{\partial u_2}(A_2h_3h_1)+\frac{\partial}{\partial u_3}(A_3h_1h_2)\right\}$$

$$\vec{A} = \nabla\varphi = \vec{e}_1\cdot\underbrace{\frac{1}{h_1}\frac{\partial\varphi}{\partial u_1}}_{A_1}+\vec{e}_2\underbrace{\frac{1}{h_2}\frac{\partial\varphi}{\partial u_2}}_{A_2}+\vec{e}_3\cdot\underbrace{\frac{1}{h_3}\frac{\partial\varphi}{\partial u_3}}_{A_3}$$

とおけば

$$\boxed{\nabla\cdot\vec{A} = \nabla^2\varphi = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\varphi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\varphi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\varphi}{\partial u_3}\right)\right)}$$

$$(u_1\ u_2\ u_3)\rightarrow(r,\theta,\varphi)$$

$$h_1=1\quad h_2=r\quad h_3=r\sin\theta\qquad h_1h_2h_3=r^2\sin\theta$$

$$\boxed{\nabla^2\varphi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\varphi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right)+\frac{\partial}{\partial\varphi}\left(\frac{1}{\sin\theta}\frac{\partial\varphi}{\partial\theta}\right)\right]}$$